Contexts Free Productions Tests and Geometry in Realistic Mathematics Education

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0 Preface

'Realistic' as connotation of mathematics education provokes confusion. On the one hand 'realistic' refers to the objectives with respect to their realizability both for the teachers and the learners. On the other hand 'realistic' concerns the didactics of (re)construction. In the latter meaning this not only means establishing the connection between reality and the mathematics to be learned, but also creating the possibility for the leraners to construct a mathematical reality.
This mathematics bears meaning because of the way it is brought forward. In this manner mathematics comes to life for the learners and stays meaningful for them even when they grasp the level of formal subject systematics after a process of abstraction. In order to enable them to do so, much attention is paid to the development of models and schemes. These and other mathematical tools encourage a process of progressive mathematisation and promote the vertical coherence in the system to be built.
Now then, mathematics education that claims 'realistic' as its main qualification possesses some specific characteristics with respect to teaching and learning.

In the first chapter these tenets will be elucidated in brief.
In the following chapters some of these tenets will be elaborated more profoundly.
Subsequently it concerns the role and function of context-problems (ch.2), special assignments like doing free productions (ch.3) and, because realistic mathematics instruction has its special demands for it, the question of testing (ch.4).
A view on realistic geometry will close our discourse (ch.5). Here again the theoretical framework will be dealt with and reflected upon.

By doing this the circle becomes closed. This, however, does not mean that realistic mathematics education behaves like a vicious circle. On the contrary we would rather say. This is proved by the fact that the awareness becomes more and more ubiquitous that schoolwalls may not be barriers between the mathematics of the outside world andthe inside mathematics anymore.

At the eleventh conference of the North American chapter of PME (Psychology of Mathematics Education) at New Brunswick NJ 1989 we explained earlier what has been put in order neatly in this book.

State University of Utrecht, The Netherlands,

Koeno Gravemeijer
Marja van den Heuvel
Leen Streefland

1 Realistic Mathematics Education (RME) What does it mean?

Leen Streefland

1.1 Introduction and overview

Learning mathematics, such as considered in this contribution, means *constructing* mathematics or – to say it more explicitly – proceeding from one's own informal mathematical constructions to what could be accepted as formal mathematics. This implies two questions to be answered:

'What are the features of this kind of learning mathematics?' and
'How to realise it?'

When switching from learning to teaching one might correspondingly ask:

'What are the background principles of RME?' and
'By what teaching methods can this kind of learning mathematics be brought about?'

Both questions will be answered in § 1.3 by the paradigmatic *example* from initial learning of subtraction (< 100), presented in § 1.2. In § 1.4 our approach will be compared with that of Thornton a.o.1983, where some general characteristics of both approaches will be summarised and considered theoretically.

1.2 Subtraction (<100)

There are many applications of subtraction in which column subtraction is neither the most natural nor the closest fitting approach. More often than not do children prefer quite different strategies. Let us consider the following

Example:
My book counts 53 pages. I have read 26. How many more shall I have to read to finish it?

Many children don't identify this problem as a subtraction. As a matter of fact, three out of four pupils in the intermediate grades do not. (Cf. Treffers a.o. 1989. So they are not able to apply:

$$\begin{array}{r} 53 \\ \underline{26\ -} \\ \ldots \end{array}$$

in this particular situation. Even if they can they are handicapped by other awkward features of algorithmic column subtraction: To half of the pupils in the intermediate grades the algorithm itself is problematic; by the 'right-to-left method' any feeling for number size gets lost. The algorithm is artificial, it does not match strategies of mental computation and estimation; it does not respect strategies of two-way complementary counting, and the structure of the algorithm itself deviates from that of many applications. Even the simultaneous use of con-

crete material creates specific problems as has been
ver if lost, this specific algorithmic ability is not ea
Omanson 1987). So we have to answer the questio
from a realistic point of view, that is in a (re)constr
Well, let us return to the *example*. Even though they
as a subtraction many children are able to solve it. I
they try to bridge the distance from 26 to 53 by mean
plementary counting).
E.g.

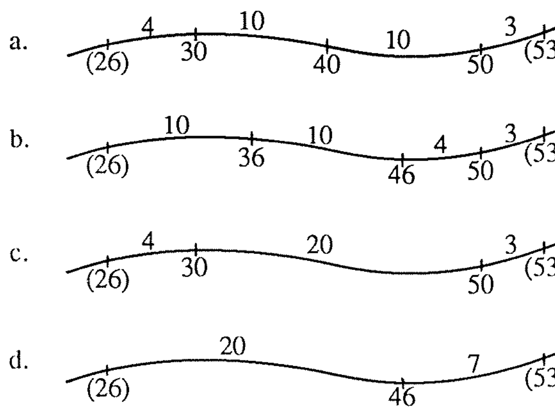
a. (26) + 4 (30) + 10 (40) + 10 (50) + 3 (53)

b. (26) + 10 (36) + 10 (46) + 4 (50) + 3 (53)

c. (26) + 4 (30) + 20 (50) + 3 (53)

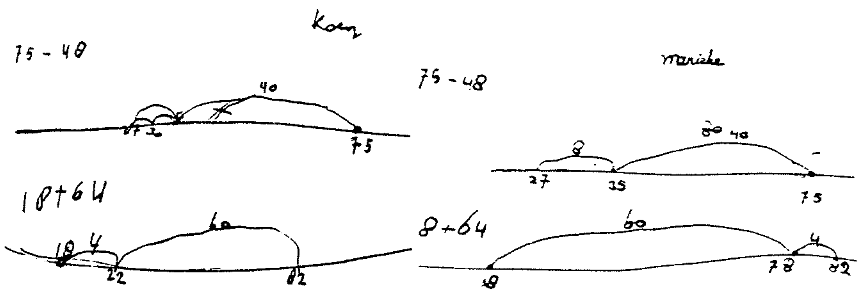
d. (26) + 20 (46) + 7 (53)

These various methods can be represented on the n
an empty or yet unstructured numberline.[1] Let us
afore mentioned strategies, children fix key-points



Depending on the choice of key-points, solutions m
and so on). In teaching, the drawbacks of the trad
wards subtraction can be made good for by the use
line. Indeed, working on the empty numberline is
pound and with two-way complementary counting
the mirror image of addition in counting onwards st
variegated methods of pure and applied subtraction
guished in the program:

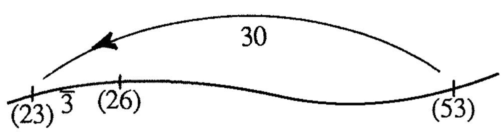
1. Working by *rows* according to different counting and compounding methods or by mental complementary computation as shown before. Specimens of pupils' work:



2. Working by *columns*, albeit by means of different methods of mental computation both according to rows and columns, for instance:

a.

53		53	
<u>26 -</u>	or	<u>26 -</u>	
30 - 3 = 27		30 - 3	
		<u>27</u>	



This is kind of conventionalised or stylised mental computation.

b.
53
26 -
$3\bar{3}$ (a debt of 3 on a possession of 30 makes 27).

3. If required a final algorithm can be aimed at, respecting the children's predilection to work from left to right as well as the historical roots of the algorithms.[2] Experiments on learning long division in stages has shown how final algorithms, in order to be mastered, can be headed for by slow and gradual steering.

1.3 Theoretical framework of teaching-learning principles.

In a way the above *example* is a concrete form of the fundamental and mutually connected principles of teaching and learning mathematics according to the didactics of reconstruction such as formulated by Treffers (1987). Of course this mutual connection between teaching and learning principles does not exclude other links.

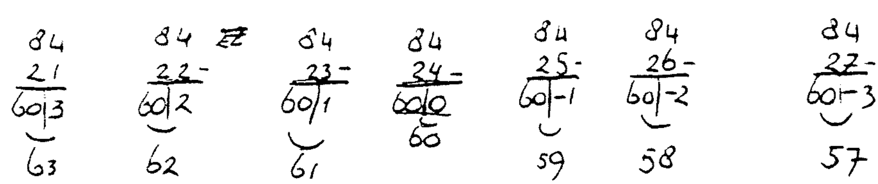
The principles (or tenets) to be considered are:
1. Constructions stimulated by concreteness.
2. Developing mathematical tools to move from co
3. Stimulating free productions and reflection.
4. Stimulating the social activity of learning by int
5. Intertwining learning strands in order to get m
tured.

1.3.1 Constructions stimulated by concreteness.
Pupils find out their own procedures to compute 53
base of orientation for the operation involved. All k
situations are starting points for the children in th
process directed towards the mastery of these pro
like counting, compounded counting, and structurin
tion, which from the beginnings onwards are firmly
The above *example* in particular and real problems
opportunities for the children to realise what numbe
to.
To stress it once more, concreteness in the RME se
but materials like (Dienes) blocks to be manipula
mean using suitable contexts (cf. chapter 2). This, t
construction principle: by lending concreteness to a
into mathematics, as well as through mathematics, w
er way to learn mathematics. Reality both serves as
operations and structures and as a domain to apply
has been shown – 'realistic' means adapting the con
ren's informal strategies.

1.3.2 Developing mathematical tools to move from
Learning a mathematical concept or skill is a long-te
through different levels of abstraction. Opportunitie
learners to proceed along levels of increasing abstr
tions of the above *example* reflect this phenomenon
reading the book, and most strongly those which re
complementary counting. Computing is still a con
tached from the context. Later on when the numberl
metic are used, reference to the concrete can still
more needed to apply the notations. The empty nu
thinking model with respect to applications and as
third or abstract level, when the procedures of me
been formalised, the procedure will entirely be und
the formal number system. There the depth structur
clear. This is the reason why one should insist on th

distinguish three stages in the teaching-learning process. Let us add that intertwining real problems with learning subtraction guarantees a larger measure of applicability of the operation involved.
Mathematical aids and tools like the unstructured numberline are needed to achieve the required raises in level. By means of these tools pupils move on their own from the concrete via the intermediate to the abstract level while, to be sure, guided and supported by their teachers. In other words, by means of these tools the pupils will be able to understand their provisional mathematics as derived from the concrete sources in the intermediate stage between the concrete level and the level of formal subject systematics, rather than being confronted with the latter from the start onwards.
Obviously, in no way are our levels related to the well-known tripartite sequence: concrete material, pictures, mental actions (or, in Bruner's terminology, enactive, iconic, symbolic). Instead our levels refer to the degree of closeness to context problems, allowing for primitive or informal strategies, and to moving onwards to more formalised procedures within the systematics of the subject area. Indeed, the intermediate level is of a paramount importance. Aided by tools to be developed in the teaching-learning process, the pupils can invent and develop their informal strategies.The empty numberline is such a tool, which functions as a bridge between the concrete and the formal level.
'Concrete - abstract' is of course not an absolute distinction. Abstract knowledge of natural number, for instance, is one of the concrete sources of algebra. So the third or abstract levels in lower order learning processes will serve as concrete levels in higher order mathematical activities (Treffers, 1987; Streefland, 1987).

1.3.3 Stimulating free productions and reflection.
Learning mathematics and especially raising levels is promoted by reflection, which means thinking about one's own thinking. As an example let us consider the development of an algorithm out of mental column subtraction using the 'possessions and debts' model (cf.section 1).



For children it is quite natural a method (cf. Labinowicz, 1985), and at the same time an opportunity to invent their own appropriate notations. By gradual shortening this can eventually lead to what rightly deserves the name of an algorithm for subtraction:

a) 8371
3754
5000 - 400 + 20 - 3 = 4617

b) 83
37
50
-4
-
-
46

c) 8371
3754
$5\bar{4}2\bar{3}$
$462\bar{3}$
4617

For the teacher it means giving the learners continua
spective views on what they have done and to stim
volves anticipation on what will come next in the co
Both 'free productions' – to be defined and exempl
in chapter 3 - and cognitive conflicts are important n
the requirements for reflection as the driving force
esses can be met by selecting special assignments
promoting the pupils' own productions.

1.3.4 Stimulating the social activity of learning by i
The next principle of instruction asks for learning as
of fact, learning takes place in the social school envi
ing to be restricted to a purely individual concern!
Learning subtraction in a reconstructive way aims a
struct and produce problems and mathematical proc
realities and developing various techniques. Since
evokes differentiated and variegated patterns of rea
above specimens of children's work, there are recu
action. Pupils will compare and exchange ideas, disc
different levels of mathematisation and negotiate
progress. By reflecting on their approaches they wil
methods of visualising and schematising, for instan

- translating numberline procedures of complementary counting into computation methods;
- transposing row procedures like those on the numberline into column procedures;
- grasping the 'possessions and debts' model;
- hitting on particular notations to curtail the previous approach;
- looking for each other's individual productions.

In general, opportunities to compare one's own work with that of others function as well as opportunities to reflect on one's own methods of problem solving. The requirement for interaction in order to provoke group learning can be met by respecting the learning environment at school as a social context for learning. It is the place where the exchange of ideas, negotiations, refutation of arguments, discussions and so on can serve the long-term learning process, and the prerequisite for truly individualized teaching. At the same time interaction helps the entire group to progress in learning mathematics. Rather than cutting off possibilities to get informal strategies and procedures developed by the pupils, they are accepted, promoted and exploited.

1.3.5 Intertwining learning strands in order to get mathematical material structured

The final principle of instruction concerns intertwining learning strands. Like in the case of subtraction, from the very beginning onwards, related parts of the programme are to be intertwined in order to have operations well enough understood to be applied successfully.

It has been argued that children who are learning the main operations with natural numbers are inclined to apply procedures evoked by the context. In other words, while applying methods, they still remain committed to the context. By our approach informal procedures of addition and subtraction such as counting onwards and backwards, involving mental computation and estimation, and finally the development of an algorithm will be strongly interconnected and interwoven.Rather than storing away in one's mind a collection of detached elements of knowledge and skills, learning mathematics will mean constructing structured knowledge and skills fitting into a well-organised and meaningful whole. In RME the long-term perspective, such as displayed in the level structure of the learning process, is of paramount importance. If courses in different subject areas are both interwoven with each other and related to reality as much as possible, pupils will construct coherent and well-structured knowledge and skills, with a large measure of applicability.

1.4 Overview and reflection.

Our exposition reads like an implicit criticism of the one-sided approach of column subtraction supported by such concrete material as blocks. Explicitly formulated our objections are that:

- children's informal strategies are both neglected
- levels of progression as usually distinguished by ficient explanation;
- the final level of algorithmisation is headed for too straightforwardly;
- from the very beginning onwards arithmetical presented;
- algorithms to be learned are divided into partial number size (falsely identified with levels).

The main characteristics of our approach were:
- taking into account children's informal strategi rows and columns;
- providing the pupils with mathematical tools to between the concrete and the formal level (and in a different way but the usual one);
- intertwining related learning strands of countin mation, column procedures, and applications.

The plea for more mental computation in the 'NCT here by suggestions how to connect mental computa rithm(s) - suggestions lacking in the 'Standards'.
We mentioned five theoretical principles, which retu teaching-learning theories such as.[3] In our approa which differs - for instance - from that of the cogniti ematics. Our background theoretical framework is a and cognitivism. On the one hand a manifold of op ren's own contributions to the teaching-learning pr the overall goal of our approach is that pupils reach atics. Finally we like to draw attention to the way ou that is, connecting the general theoretical aspects t and confronting them with a different approach.

Notes:

1. Hassler Whitney described the idea of the unstructured numberline in an unpublished paper 'Sane decision making in Mathematics Education'.
   See also Grossman (1975), Labinowicz (1985), Madell (1985), Treffers a.o. (1989).
2. Cf. Streefland (1988) and his chapter on free productions to this book and also Treffers a.o.(1989).
3. Cf. Cobb (1987). In piagetian theory, for instance, the distinguished basic principles can be recognised, though with respect to interaction and learning in a social environment this aspect was recognised by Piaget rather late; see for instance Piaget's discussions with the journalist Claude Bringuier.

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2 Context Problems and Realistic Math

2.1 Introduction

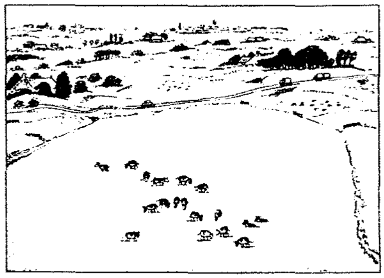
One of the main features of the so called 'realistic'
Netherlands is the way context problems are being
context problems hold a position comparable to that
Dienes blocks) in the structuralist approach - an app
tional design based on information-processing psych
role of context problems in realistic instruction let
the structuralist approaches with each other.
Structuralist instruction starts with establishing rela
cepts and procedures, before it is heading towards a
the end of some chapter it distinguishes itself from re
text problems are lacking in structuralist instruction
Indeed, the well-worn stereotyped word problems
context problems. Meant as mere applications, they
matter learned shortly before, which explains their o
set one has to resort to situations where the previousl
can be applied without difficulty. In this respect, s
long tradition in mathematics instruction.
The realistic approach rejects the instructional part
knowledge and its application. On the contrary, ap
role in the factual development of mathematical con
tributed in structuralist instruction to concrete mode
ing is played by context problems in realistic instru
In our exposition this will be illustrated by pictures
tiplication and division. For a start, however, let us t
to explain what we mean by the word 'context'.

Ten birds on a tree. Two of them are shot. How

The joke is meant to fool those who answer 'eigh
since all of them are frightened by the sound of the g
than the explicit data of the problem; it embraces
calls up. This property characterises context proble
allowed to bring to the fore their own knowledge ab
need this common knowledge for problem-solving.
But there is more to it. What is the background of t
to fool people by such a simple problem? There is so
and in particular in that little world of school prob
often implicit - rules. In a Dutch textbook of the tra
following task, where students are expected to use

width' and 'two times length plus width'.

The perimeter of a meadow is 1.2 km.
The length is 400 m.
The area is ...... hm^2.



* *Think of the perimeter divided by two.*

Apparently the textbook author saw no reason to instruct the illustrator to give the meadow a rectangular shape, whereas the illustrator was not acquainted with the rule that meadows are always rectangular, at least in school problems.
'A car drives with a speed of 90 kilometres per hour from A to B...', will immediately be recognised as one out of the world of school problems. How far away from reality it is, became clear to a colleague of mine who administered the following exercise to 14 year olds.

A car is driving with a speed of 90 kilometres per hour.
How long will it need for 500 meters?

The students reacted: 'One cannot possibly tell unless one knows how fast it is going.' At first, my colleague felt being fooled, but after a while she understood that the students thought of their own experiences with trips by car. They knew that riding by car involves putting on the brakes, accelerating, waiting for trafic lights, and gaining momentum on the main road. Indeed, the statement that it takes one hour to cover 90 kilometres does not provide any information about the speed at an arbitrary moment.
The usual instruction of word problems tends to disregard the student's informal experiential knowledge. So it both blocks the way to mathematical applications and misses the opportunity to use this knowledge in a beneficial way.

2.2 Informal knowledge
Experiential knowledge like this is also known as informal knowledge or intuitive notions, which indicates that it has been gathered without teaching, or that students are hardly aware of it.
Adults also have this kind of knowledge at their disposal. Unwittingly, we know a lot more than we realise ourselves. DiSessa gives the example of the move-

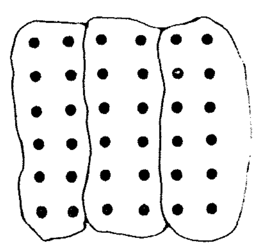
ments someone makes who lifts an empty dustbin
dustbin is loaded with stones. From the odd movem
can infer the existence of a lot of scientific knowle
that of Newton's laws.

If you can rely on informal knowledge, it is often ea
in *reality*, than it is to solve its counterpart in *mathe*
(1984) as well as others found that the developmen
to solve context problems goes ahead of that of
school. Conversely, one may expect to be able to ev
dures by administering context problems which
knowledge not yet taught at school. Let me show y
We gave the problem of dividing 36 by 3 to third-g
multiplications with numbers bigger than 10, let alo
vision procedure (Galen, e.a.; 1985). Of course, rat
resentation like 36 ÷ 3 we presented a context probl

Three children shall divide 36 sweets.
How many sweets will each of them get?

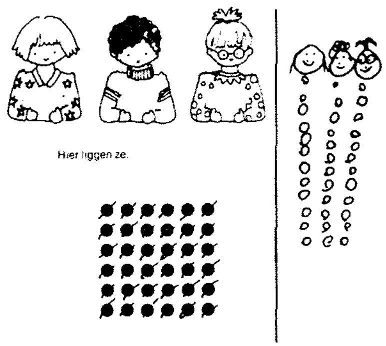
The students invented all kinds of solving procedur

- dividing on a geometric base:



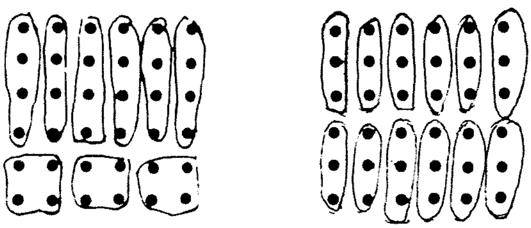
Thus the area of a square containing the 36 sweets is divided in three equal parts.

– distributing one by one:

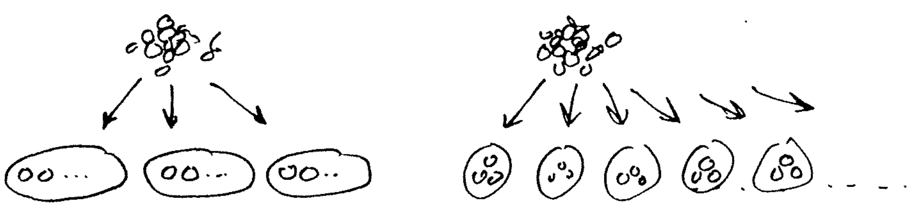


Thus the sweets are distributed one by one. One by one they are crossed out of the total and added to one of the rows. (The students even tried to copy the children's pictures faithfully.)

– grouping:



Some students draw equal groups, divide them by three and count their contents. Some of them draw groups of three: each time one sweet is distributed to each of them, the stock diminishes by three.
Two main solving procedures show up: distribution division and ratio division. The distribution division appears most clearly in the geometric solution, where the student interprets the problem by the way of creating three equal groups.



distribution division *ratio division*

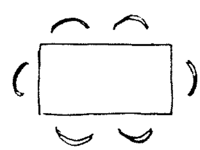
Some students ask the different question 'how m
made?', which aims at a ratio division.
The relation between both of them was already in
one focuses on creating three equal groups. At the s
ry time each one gets a sweet, the original numbe
three. Therefore, this approach causes the question
can be repeated. It requires, however, a translation o
inal context problem aims at a pure distribution divi
suggested by a problem like the following[1]:

A net keeps three balls.
How many of such nets will be needed for 36 ba

2.3 Developing long division

For two reasons we now stress ratio division. First,
connected to the column algorithm. Second, traditio
it difficult to relate this algorithm to that kind of ap
(1981) shows that, rather than using the column alg
to solve applied division problems by repetitive su
ample (Dolk & Uittenboogaard; 1989) shows how
voked by ratio division. Children of about the same
were asked to solve the following problem.

Tonight 81 parents will visit our school
At each table six parents can be seated.



How many tables do we need?

The students produce all kinds of solutions:

– some use repetitive addition: 6 + 6 + 6 + ..., or st
  ably based on addition, 1 x 6, 2 x 6, 3 x 6, ..., som
  ing sequence 6, 12, 18, ...;

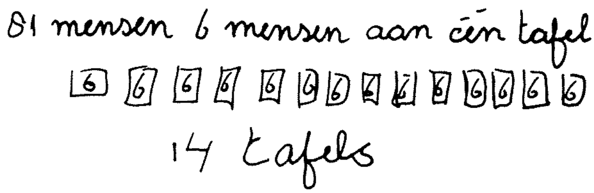
- some use 10 x 6 as a starting point, in order to continue by multiplication or repetitive addition;
- one student knows 6 x 6 = 36 by heart, which is doubled to get 12 x 6 = 72 , one 6 is added, and finally one more 6.

The teacher stimulates the students to compare their solutions. Obviously most of them find the first jump to 10x6 a handy trick. When afterwards a similar problem (concerning the same night at school) is administered, it appears that a substantial part of the students imitate the 'ten times' trick spontaneously.

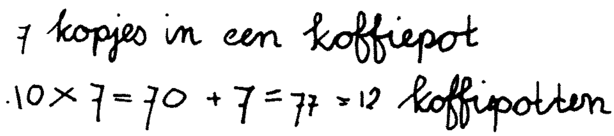
One pot holds seven cups of coffee, every parent gets one cup.
How many pots of coffee are to be brewed for the 81 parents?

Only one child sticks to the single step method. Thirteen use the 'ten times' trick, compared with six in the first round. From the work of three children it not clear how they got their answer.

The solution procedures of one of the students.
81 people; six at each table (tafels)



each pot holds seven cups, how many pots (koffiepotten) are needed?



It should be noted that the teacher had not made any suggestions whatsoever to use the 'ten times' trick. Rightly one had expected that those who saw its advantage and trusted it, would use it on their own initiative.
The procedure that is here employed to solve what in principle is a division problem, can be labelled as 'compounding'. One tries to approach the dividend as closely as possible by adding up multiples of the divisor. As a matter of fact, we ourselves prefer this strategy for mental divisions. For instance, the easiest way to know the average gasoline consumption of one's car is to reset the hodometer when the tank is full, and to compare the number of kilometres with the amount of fuel needed to refill it next time at a gas station. One may even try a more precise estimate, while driving away from the gas station. In our country gas consumption is measured by kilometres per litre of fuel. Suppose you used 34.09 litre for 466.8 km, which would require the division 466.8 ÷ 34.09. To keep it sim-

ple let us do 467 ÷ 34. Obviously 34 goes at least te
34 gives 340 to start with. Two more times?
No, three more times gives 340 + 102 = 442, whicl
more precisely, one decimal at least: with 25 left by
0.7 x 34 ≈ 17 + 7 = 24. So our estimate would be 13
If we compare this calculation with the column al
mental arithmetic resembles the standard procedure.

```
34 / 467 \  13.7
     34 -
     127
     102 -
       250
       238 -
        12
```

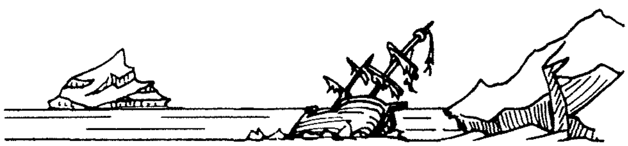
In fact, translated into the mental procedure, this me

```
34 / 467.0 \10+3+0.7=13.7
     340 -    [10 x 34]
     127
     102 -    [3 x 34]
      25.0
      23.8 - [0.7 x 34]
       1.2
```

However, the algorithm is so condensed that one ha
step 10 x 34 = 340 rather than a mere 34 is subtracted
so difficult to recognise the underlying repetitive su
dure: after any subtraction of a multiple of the diviso
is left. In fact the column algorithm of long division
breviated manner to perform a division by counting
subtracted from the dividend.
In realistic mathematics instruction it is tried to teac
letting it evolve from informal ones in a learning pro
ation where the tool of repetitive subtraction offers i
the start onwards rather large numbers can be allov
which seems to be an advantage.
The broader context of the following problem is a st

got their ship stranded at the isle of Nova Zembla.

NOVA ZEMBLA



The captain of the stranded ship is told that there are 4000 biscuits left.
The crew consists of 64 members. Each man gets 3 biscuits a day,
which means 192 biscuits a day the whole crew.
How long will this supply last?

We can almost see the supply of biscuits diminish day by day, every time a ration is consumed.
What makes this problem interesting is the variety of solving procedures on different levels. The students would not stick to subtracting 192 once at a time. They would use multiples of 192 as well, say, decuples, or doubling.

4000		4000		4000	
192	- 1 day	192	- 1 day	1920	- 10 days
3808		3808		2080	
192	- 1 day	384	- 2 days	1920	- 10 days
3616		3424		160	
192	- 1 day	768	- 4 days		
3424		2656			
192	- 1 day	1536	- 8 days		
etc.		etc.			

With an appropriate context problem one may induce children to using decuples.[1]

1128 supporters want to visit the away football game of Feijenoord.
The treasurer is given the information that one bus can take 36 passengers
and a reduction is obtained for every ten buses.

The information on the reduction may work as a suggestion to calculate the number of reductions. It will call the students' attention to the opportunities offered by the decimal system. Even then various solut

36 / 1128 \		36 / 1128 \	
<u>360</u> -	10x	<u>360</u> -	10x
768		768	
<u>360</u> -	10x	<u>720</u> -	20x
408		48	
<u>360</u> -	10x	<u>36</u> -	1x
48		12	
<u>36</u> -	1x		
12			

Such leads on the way to the column algorithm are o
to be made by the students at their own level, for bui
riential knowledge and performing short-cuts at thei
Working with real problems also implies a different
the remainder, that is, as a real life phenomenon that
rather than as a peculiarity of non-terminating divisio
ed for by formal arrangements. If the context is taken
is not an acceptable answer. What can we do with
there are several possibilities, distribute them over th
bus, or speculate on the withdrawal of at least 12 at t
The realistic approach makes us aware of the variety
attributed to the remainder. Treffers (1989) listed the
division 26 ÷ 4:

1. *One has to transport 26 persons by cars.*
   Each car takes 4 passengers.
   How many cars will be needed?
2. *A rope of 26 metre is cut up in pieces of 4 metre.*
   How many pieces does one get?
3. *If 26 bananas are fairly to be divided among 4 p*
   how many bananas will each of them get?
4. *A walk of 26 km is divided in 4 equal stages.*
   How long is each of them?
5. *A rectangular pattern of 26 trees with 4 trees a r*
   how many rows will there be?
6. *A rectangular terrace with a size of 26 square m*
   has a width of 4 meter.
   How long is this terrace?

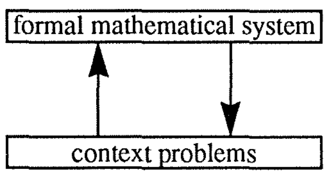
The interpretation of the remainder largely depends o
result of 26 ÷ 4 has to be used. The large variety of ap

the question of how to take care of the applicability of skills and concepts in mathematics instruction.

2.4 Two models

The fundamental change brought about in mathematics instruction by the realistic approach, is most apparent in the way applications are dealt with. The usual view on mathematics is that of the ready-made system with general applicability, and on mathematics instruction as falling apart into learning the formal mathematical system and learning to apply it. For the realistic approach the emphasis is on mathematising, mathematics viewed as an activity, a way of working. Then learning mathematics means doing mathematics, of which solving real life problems is an essential part. Manifold context problems are integrated in the curriculum from the start onwards.

The two fundamentally different views on mathematics and mathematics education imply essentially different mathematical learning processes. With mathematics as a formal system, its applicability is taken care for by the general character of its concepts and procedures, and thus, first of all, one has to adapt this abstract knowledge to solving problems set in the reality. One has to translate real life problems into mathematical problems. Let us visualise this as follows:



The model describes the process of solving a context problem with the help of the formal mathematical system. First the problem is translated; it has to be formulated in mathematical terms, as a mathematical problem. Next, this mathematical problem is solved with the help of the available mathematical means. And at last the mathematical solution is translated back into the original context. Many aspects of the original problem may have been obliterated when it was transformed into a mathematical problem. Although some of them might be mere emotional. It can also happen that the original problem does not allow for the exactitude which is suggested by the mathematical solution.

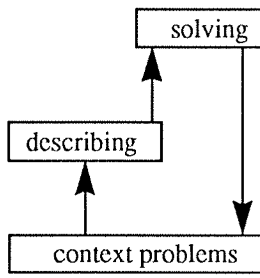
On the whole the 'translation' described above boils down on recognising problem types and establishing standard routines. As soon as we choose to teach 'mathematics as an activity', problem-solving gets another meaning. It becomes problem-centred, that is, rather than using a mathematical tool, the problem is the proper aim. Although, even if interpreted as an exploration, problem-solving passes through the same three stages of

- describing the context problem more formally;
- solving the problem on this (more or less) formal level;

– translating the solution back;
the character of these activities is now fundamentall
ing to *fit* the problem into a pre-designed system, on
to come to grips with it, and this happens in particul
and by means of identifying the central relations in t
than using commonly accepted mathematical langu
sketchy, using among others self-invented symbols

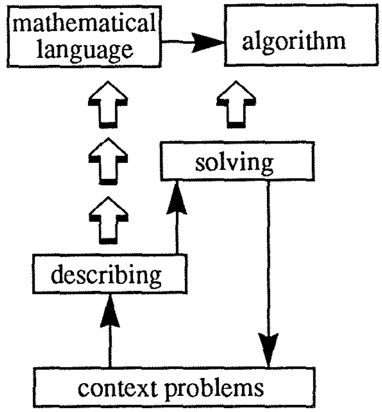
A student's description of the number of tables

The description does not automatically answer the
the problem by describing relations and distinguish
nor importance. Solving the problem as it is stated a
el differs greatly from applying a standard procedu
solving as well. Translating the final solution doe
translating a solution which is produced by a standa
and interpretation are now easier since the symbols
lem-solver, who is the one who gave them their me

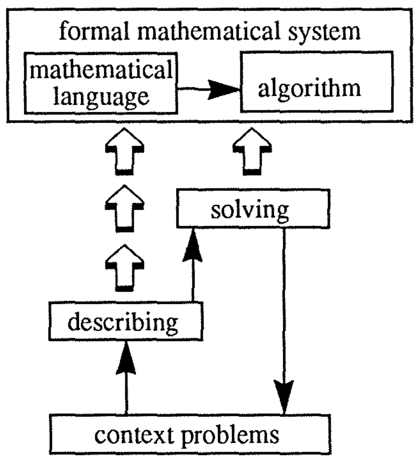


An instructional programme, full of this kind of pr
nity for students to learn mathematising context pro
Numbers of similar problems in a row will evoke a
scriptions may develop into an informal language, w
a more formal standard-like looking language, than
and formalising. This is again a process of mathema
a longer period of time. A similar thing happens to
long run solving some kind of problems may becor

condensed and formalised in the course of time. In this way genuine algorithms can take shape.



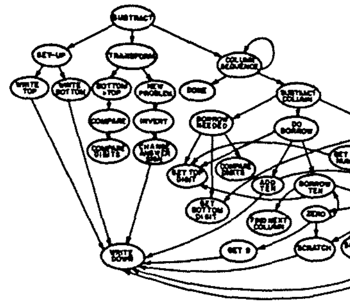
This then is a learning process by which the formal mathematical system itself can be (re-)constructed.



In Treffers (1987) the latter process, focusing on the mathematisation of mathematical matter, is called vertical mathematisation, as distinguished from horizontal mathematisation, which is mathematising context problems.
These two basic models, the 'formal model' (the one with the applications afterwards) and the 'integrated model', may be used to compare the American information-processing approach with the Dutch realistic approach.

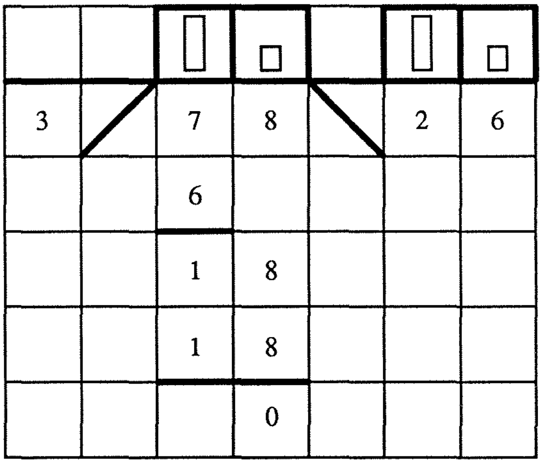
2.5 Information-processing

The ideas presented by information-processing ps
Greeno, and others perfectly match the 'formal mode
the instruction focuses on the concepts and skills as d
in their most sophisticated form. A refined form of tas
the learning task into smaller components, in order to
ess. In information-processing theory one tries to des
description of solving procedures, appropriate to sim
haviour by computer programmes. These models, how
to be implemented in instructional programmes, thoug
structional design.

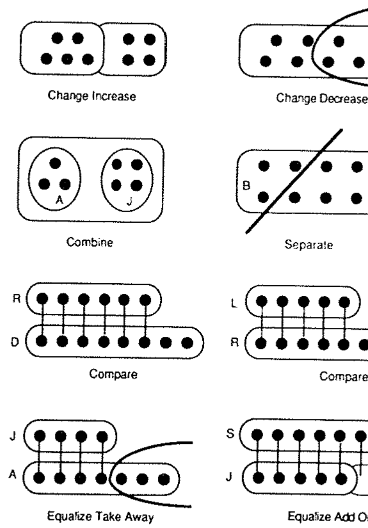


Resnick, for instance, was inspired by the models
(Resnick & Omanson; 1987). Refined models may dr
to tacit knowledge which might have been overlooke
Most of the actual instructional designs based on info
are making use of concrete models. To teach the colun
one almost always uses Dienes blocks.[2] This leads to
the case of long division (Borghouts-Van Erp; 1978)
resented by rods and cubes. The children divide the n
2 rem.1). The remaining rod, or rods, (1) are change
with the cubes that have been present before (10 + 8 =
of cubes is divided.

This procedure can be written down in the standard form of long division.



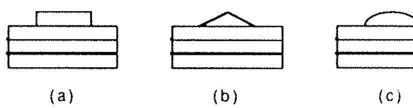
Obviously this is not a procedure that reflects many real life situations. For instance, think of the problem of the 81 parents to be seated at tables for 6. If we start representing the 81 *parents* by (the equivalent of) 81 Dienes blocks, we end up with the paradox of 13 blocks, which are now to represent 13 *tables*!
Moreover, it is not just the ratio division that causes trouble; context problems involving magnitudes fit this model as little. Since it does not cover all possible real life situations, supplementary instruction of applications becomes a necessity. Thus instruction is divided into first teaching a fixed model procedure and then teaching its applications, which strongly relies on the recognition of semantic structures in word problems.
Information-processing psychologists have listed all kinds of word problems according to skills or concepts required for their solution, in order to support the instruction of applications. Next to this, expert and novice behaviours are compared to find directions for the development of effective solving procedures. The analysis of word problems produced distinctions in semantic structures, like the well-known classification of addition and subtraction word problems by 'change', 'combine/separate' and 'compare' as main categories (sometimes supplemented by 'equalise'). One usually tries to support the recognition of the different semantic structures by diagrams. Greeno (1987), for instance, refers to Lindvall's models in the case of addition and subtraction, and to Shalin's in the case of more complicated word problems.



Diagrams for illustrative word problems in I

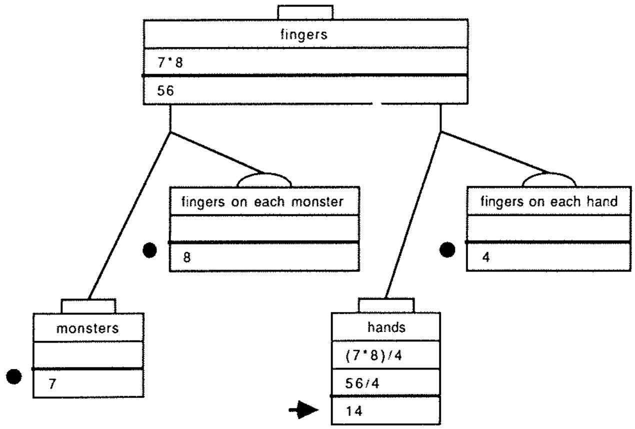
Shalin's system is based on four types of quantit
symbols, indicated as different shaped 'hats':

a. extensive quantities, as in 'Ann had 5 pencils
   hats;
b. differences, as in 'Dann has 2 more kites than R
   hats;
c. intensive quantities, as in 'Tom put 6 books
   rounded hats;
d. multiplicative factors, as in 'Sue had 3 times as
   sented by trapezoidal hats.



These symbols enable one to design representation

Dr. Wizard has discovered a group of monsters
America.
He has counted 7 monsters, and there are 8 fin
If there are 4 fingers on each monster's hand,
he find?



Curiously enough, the simplest solution is lacking: eight fingers on each monster and four fingers on each monster hand implies two hands on each monster, so there are
2 x 7 = 14 hands.
To be sure, one also tries to establish metacognitive skills and strategies. However, most of the students do not know how to apply general heuristics. Lesh's research (1985), for instance, shows that the applicability of heuristics is strongly affected by the availability of domain specific knowledge. Therefore it is not so surprising that Schoenfeld (1987), for instance, abandoned the idea of general heuristics, and changed it for the idea of 'mathematical people'. He states that one can teach problem-solving effectively by creating a microcosm of mathematical culture, which means an atmosphere where mathematics is the medium of exchange, where one talks about mathematics; explains it to one another; where one shares solution processes including false starts; discusses solutions, and so on. In other words a manner of working closely related to the realistic approach.

2.6 Didactical phenomenology

In virtue of the 'formal model' the applications follow upon the subject matter as taught by abstract reckoning within the formal system. Freudenthal (1973, 1983) opposed this order, which he calls an anti-didactical inversion How did humankind gather its mathematical knowledge? By studying reality in the first place! Mathematics developed by generalising regularities, developing and formalising notations, and algorithmising solving procedures of recurring practical problems. From applications mathematics evolved into a formal system, but at present one tries to teach children the formal system first and applications after-

present one tries to teach children the formal syste
wards – an inversion which Freudenthal rightfully
As an alternative to this inverse order Freudenthal s
one with the help of didactical phenomenology. D
scribes concept attainment, or as Freudenthal prefe
mental objects' based on a phenomenology. He de
'constitution of mental objects', which according to
ment, and can be very effective even if this final s
not taken. Didactical phenomenology presupposes
nomena which stimulate the student to form the m
This sharply constrasts with approaches where on
We just mentioned the fact that mathematics has ev
ality. Didactical phenomenology is investigating i
mathematical concepts function in reality. Context
op 'intuitive notions', which are consolidated in m
the base for the concepts to be developed. The final
established by a reinventing process.
The word reinvention refers to the idea to take t
arose in history and still arises, as a model for in
means, for instance, that it can pay to study the his
before one starts designing an algorithm course, thu
dren to develop the algorithms on their own. The a
clue: a tool to learn the column algorithms for add
we will reconsider the reinvention principle, but fi
itial phase of this process. Here context problems
tween the students' experiential knowledge and th
clarify it we choose some phenomenological aspec

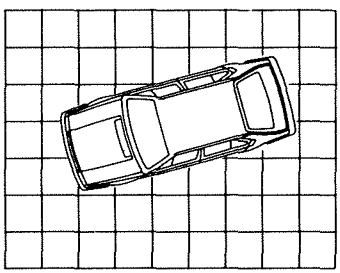
2.7 Multiplication

Didactical phenomenology strives to establish a br
skills to be developed. The need for such a broad
of long division. We elaborated on the distinction
division, and we mentioned the rectangular mode
grates both concepts. It is important for learning m
cause its validity extends to operations with magn
paradigm for the calculation of areas. It not only s
multiplication but also the acquisition of the mu
(1985) showed that most children do not bluntly m
bles in order to know them by heart; they rather in
difficult products from easier ones. Spontaneously

- changing order [3x6=18, because of 6x3
- doubling [2x6=12, therefore 4x6=
- halving [10x6=60, therefore 5x6=

- adding [2x6=12, therefore 3x6=18]
- subtracting [10x6=60, therefore 9x6=54]

According to the realistic conception the students' own solving procedures might be useful to start learning sequences with. Often the students' spontaneous behaviour anticipates on skills and concepts lying ahead. Designing instructional programmes on the strength of observations with regard to solving procedures, such as invented by the children themselves, can be seen as a further refinement of the reinvention principle.
The reinvention principle not only recommends the history of mathematics, it also refers to a particular kind of learning: Following in a sense the historical learning path, students reconstruct the mathematics discovered that way. Of course, the idea of (re-)constructing mathematical knowledge is more fundamental than the historical aspect. Freudenthal favours the reinvention principle because it is also the way of most adult mathematicians to get familiar with new mathematical ideas. History of mathematics can help to find suitable learning paths, but as now understood, the students' self-invented solving procedures also indicate possible routes. In the case of fractions, this approach was successfully elaborated by Streefland (1988).
To stimulate the development of such strategies as discerned by Ter Heege, one can employ context problems based on the rectangular model such as, for instance[1]:



How many tiles are there?

This question may be posed long before the students understand 9 x 8 as a multiplication. All kind of solution procedures will be put into play:
- just counting the tiles (including the invisible ones);
- compound counting: 9, 18, 27 .., or 8, 16, an so on;
- doubling: calculating one row, two rows, four rows, ...;
- subtracting 8 from 10 x 8.

By its reference to tiled pavements, counting tiles is a realistic problem, and meaningful for students. That is one side. On the other hand the rectangular

model as concretised by the tiled pavement provid
ular meaning - one particular meaning, which has
other situations. Yet here we are concerned with t
the rectangular model by the students. It is the way
bridges the distance between informal knowledge
The afore mentioned informal strategies may evo
arithmetic involving larger numbers. In the end thi
rithm, which may be seen as an organised form of
tion, with short-cuts by multiples of ten and hund
gorithm course for multiplication:

26 x 74 = 74
74
74
74
74
74
74
74
74
74 ------- 740
74
74
74
74
74
74
74
74
74
74 ------- 740
74
74
74
74
74
74

To learn the algorithm for multiplication (as well
order repetitive operation is chosen to start with
sidedness than in the case of a structuralist approac
have already become familiar by themselves with
cation, and secondly, because they may always
context-bound solving procedures. The realistic a
standard procedures, neither in the beginning no
quence.

2.8 General attitude towards applied problems

In realistic mathematics instruction the domain spe
tion as a base for the learning process. Semantic s

cognitive psychologists, is not ignored. One does not, however, value their explicit recognition as a prerequisite for applying a standard procedure. One relies on phenomenological structures as a means to simulate domain specific solving procedures. Only after a while will students learn that these procedures are interchangeable. Meanwhile the students are stimulated to improve their own solving procedures, to shorten, to schematise and to generalise them. In this way the final goal, a standard procedure, is reached, and the applicability of this knowledge is guaranteed.
However, there is more to it. Not only is a solid base for concept attainment laid by this approach, it also helps generate a general attitude towards applied problems. While heuristics and metacognition promote the application of some algorithm (or a combination of algorithms in the information-processing approach), the realistic approach consequently capitalises upon the students' own ideas. So the students will develop the attitude to consider it as self-evident to have a try, to use one's head, and to look what can be done with the domain specific knowledge available.
Realistic mathematics instruction implies estimation wherever it is meaningful, and the use of common quantitative knowledge about reality, for instance, to uncover errors in cuttings from newspapers. As an example take the journalist who, looking for a fair method to compare the number of Olympic medals earned by several countries, performs some calculations and commits some errors!

> Let us restrict ourselves to the Netherlands, with about 14 millions of citizens, against the USA with more than 3 billions (3×10^9), which is roughly two hundred times as much. The area of the Netherlands is something like 40,000 square metres, against the USA's 33,000 square kilometres, nearly a thousand times as much.

Though readers might be expected to know that, though small, the Netherlands is not as small as two football-fields, that the number of citizens of the USA cannot possibly be three billions as there are no more than five billion people on the whole world, and how to convert m^2 into km^2, there will not be many who pay attention to such errors. Indeed, it requires an investigative attitude to look for them, which unfortunately is undermined systematically by standardised mathematics instruction. It is part of what Schoenfeld (1987) refers to as the 'hidden curriculum': Students get a wrong idea of what mathematics consists of, and how to solve problems. As an example he uses a problem (not unlike ours on buses) set in the NAEP secondary mathematics exam:

> *An army bus holds 36 soldiers.*
> *If 1128 soldiers are being bused to their training site,*
> *how many buses are needed?*

Seventy percent of the students chose the right algorithm and performed it correctly. However, on the question how many buses are needed,

29 % said: '31 remainder 12',
18 % said: '31', and
23 % gave the right answer: '32'.

Most of them just picked the numbers from the tex
wrote the answer down, without bothering wheth

2.9 Constructivism

There is some resemblance between the realisti
nomenology on the one hand and constructivism
between both of them may be useful: on which
which do they disagree?
Philosophically constrcutivism is based on the id
something like an 'objective reality'. As a matter
ity consists of theories about reality, which notwit
herence and logic do not imply their own truth
structivists consider 'misconceptions' or 'alternat
sequences of alternative theories.
Against this background the use of concrete mat
for instance, insists on distinguishing between an
of view'. One should try to look through the stu
tations remain problematic as long as children ke
terial, rather than such as intended by the adult,
mathematics. According to Cobb, the trouble is
'representation': both mentally, that is, in the st
that is, the concrete material. The lack of a sha
them is symptomatic for the partiality of adult obs
the mental representation, are able to 'see' it in t
ports an experience of Holt's, who at the start wa
senaire rods. The striking relations between this
bers would allow a beautiful entrance to the world
ever,

> The trouble with this theory was that Bill and I *alre*
> bers worked. We could say, 'oh, the rods behave j
> hadn't known how numbers behaved, would look
> find out? (Quotation Cobb; 1987.)

Unless internal and external representation are di
mixed up: the mental representation which has t
order to mentally interpret the concrete represent
The idea to distinguish between the actor's and
inspired by constructivist epistemology. The vie
own theories about reality, led Cobb to realising
fers from the student's. Children build their own t

stood this way, realistic instruction theory may be seen as its complement. The idea of reconstruction of mathematical knowledge unites both of them. Realistic mathematics instruction may meet Cobb's fundamental criticism on the implicit observer's point of view. The ideas of the student are respected by taking their informal solution procedures seriously. This seems to be the proper way to prevent 'misconceptions'. However there will always be a certain tension between 'following' and 'guiding'.

It is at this point that constructivism and realism differ. Constructivism in the sense of Confrey (1987) is essentially a kind of appreciation of theories, such as developed by young children. As far as comments on instruction are made, it seems that an epistemological and a normative point of view are interconnected: the students' theories should be valued as equal to the teacher's, which implies that students be free to choose the direction in which they tend to develop.

Realistic educators are more careful, they do try to steer the theory-building of the students. By means of appropriate assignments desirable developments are provoked, adjustments take place thanks to social interaction and discussions, as well as by means of context problems that lay bare the weak points of alternative theories. In other words, realistic mathematics instruction focuses on carefully planned long-term learning processes where one tries to do justice to the students' ideas.

2.10 Conclusion

As a conclusion let us sketch the realistic approach to the main issues in instruction theory:

- First, applications, domain specific knowledge and strategies: Applicability is inherent to the curriculum by virtue of the stand on context problems. Let us add that in the measure students feel at home in the mathematical field, mathematics itself becomes a context!
- Second, models: Rather than being offered right away, they arise from problem-solving activities. So they can function to bridge intuitive notions and formal mathematical objects.
- Third, construction: With regard to the question whether students should construct their own knowledge or whether one should rely on explicit instruction and feedback, realists take the position that the construction process can and has to be guided by means of special assignments, free productions and interaction.
- Fourth, social versus individualised learning: The question may be answered by a reference Schoenfeld's idea of 'mathematical people'.
- Fifth, complexity of learning: As opposed to tasks analysis, the approach is rather holistic. As reflecting the complex reality, context problems are to get learning strands intertwined, which also serves the development of intelligible solving procedures and relational understanding.

Notes:

1. Example(s) taken from a Dutch textbook series: Grav
2. See also Labinowitz (1985) for problems with Dienes

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3 Free Productions in Teaching and Learning Mathematics

Leen Streefland

3.1 Introduction and survey

The question I wish to tackle in the present paper is[1]: How to influence children to produce by themselves – albeit under guidance – their mathematical abstractions. (cp. Cobb, 1987.)

In order to answer this important question I will deal with successively:

- children's own - free - production in mathematical instruction-what does it mean?(3.2);
- functions of their own production in the teaching/learning process, with examples (3.3);
- own productions in developmental research after reconstructible instruction (3.4).

A brief reflection will conclude the exposition (3.5).

3.2 What is own production?

In productive mathematics education children, guided by their teachers, construct and produce their own mathematics. The pupils' mathematical activity expresses itself in their construction and in the production resulting from reflection on the constructions. Treffers (1987, p.260) has introduced this distinction, which according to himself is no matter of principle. Free production is rather the most pregnant way in which constructions express themselves. What, however, is own production? In order to answer this question we shall look out for the preconditions and circumstances under which productions emerge or may emerge in instruction.

By *constructions* we mean:

- solving relatively open problems which elicit- in Guilford's terms - divergent production, due to the great variety of solutions they admit, often at various levels of mathematisation; and
- solving incomplete problems, which before being solved require self-supplying of data or references.

An example of the first: How to divide two bars of chocolate among four children?

An example of the second: A radio message on a 5 km queue at Bottleneck Bridge: How many cars may be involved?

The construction space for free *productions* might even be wider:

- contriving own problems (easy, moderate, difficult) as a test paper or as a problem book about a theme or for a course, authored to serve the next cohort of pupils.

An example, say, for grade one: Think out as many sums as you can with the result five.

Finally there are border problems, that is, of cons
strong productive component, which require devis
notations, schemes, or models. In our illustrating
various functions own productions can have in the
well as in research).
In fact, a production problem can involve more tha
division according to functions is again a matter of

3.3 Functions of own production in the teachin

3.3.1 Preliminary survey

If children's learning is to be expressed in their ow
tions have to be viewed under the aspect of instruct
didactical value (though of course from the learne
completeness, we will distinguish the following fu

- grasping the connection between phenomena in of description and organisation (horizontal mat
- seizing the opportunities of continued organisi matical material (vertical mathematising) (3.3.
- uncovering learning processes, and reversing w
- producing terminology, symbols, notations, s both horizontal and vertical mathematisation (

Each of these functions will be illustrated by exam
cases it will appear, that being productive in the
both reflection and anticipation on the teaching/lea
The various functions will finally be considered
course construction and developmental research (
Finally some remarks will be made (3.5).

3.3.2 Grasping problems

Example: 'The size of The Netherlands' (after Tre
of calculation and meansuration by estimate:

Somebody affirms that the area of The Netherland
cording to Larousse Encyclopedia, he says. What

We will give an impression of the course of a lesson with 11 to 12 years olds who have received traditional (rules oriented) instruction, although in their last year (grade six) a few richer problems happened to emerge in the lessons.

The pupils start working while the teacher walks around, assists groups of pupils, and afterwards conducts the retrospective discussion.
At the start of the lesson the teacher had a brief talk with Mar:
Mar: 'Then I should first know what is a square meter.'
Mar: 'I do know that a football-ground is a hectare.'
T: 'What's your size, Mar?'
Mar: 'One meter seventy.
T: 'And now a square meter. Pay attention to 'square'.
Mar: 'I see. It is four times a meter (indicating a square).
T: 'This desk, is it as big as a square meter? (in fact it is 1.30 m. by 0.70).
Mar: 'No, it isn't a square, so it is not a square meter.

Follows some explanation. Mar is progressing but time and again new obstacles arise; for instance, when The Netherlands is modelled into a rectangle of 200 km by 300, and the area should be calculated. The estimated dimensions are to the point but 200 x 300 is done by column arithmetic. Mar's mathematical activities oscillate between two extremes: intelligent estimating and thoughtless calculating.

Most of the pupils appear to know very well what size a square meter is, and understand decently what is area, but they still lack the mathematical attitude of trying a multiplication related to the given area, or starting at the other side, that is to make an estimate of the size of The Netherlands on the strength of available experience.
They reproach the teacher walking around: 'It is so big a number that one cannot imagine it, so there is nothing to comment.'
They are given a hint; the size of a garden or so. It is sufficient to put them on the right track. In due course everybody is adjusted to explore whether the given number of square meters is possible. In retrospect the pupils deliver the comment (briefly summarised):
- If it were true, hundreds of people would live on a square meter because millions are living on the approximately 37,000; so that is impossible.
- Upwards of a 36,000 square meters is like a strip long 36 km and wide 1 m, and that is rather like a path through The Netherlands.
- The given number of square meters is not much more than a rectangle of 200 meter by 180, that is about six football-grounds-it is good for Gulliver's Lilliput.
- The Netherlands is about a rectangle of 200 km by 300 (cf. Mar's estimation), thus 36,842 square meters cannot be right.

All these comments are discussed. Almost everybo
ious ways of reasoning and to compare them with
among these comments offers the teacher the oppo
origin of the error. The whole group agrees that it s
metres. Then the teacher raises the question: How c
ometres have come out?
In a final discussion objections are summarised: W
Do they belong to the given area?

What about the tides? Isn't our country larger at lo
big can the difference be? Isn't the area to be consi
Finally the crucial question: 'Wouldn't such a prec
strength of some model of The Netherlands?'
The teacher herself explains things: the fixed low
model define the calculation. In detail this model c
is how the size of The Netherlands is verified.
The foregoing was a good starting point. This is pr
concerning the classification of countries according
scored at the Olympic Games of 1984.

> The classification obtained by this method has func
> ow classification, never included into official tables
> a closer look to this equivalising formula.
> Since it requires some arithmetic, let us restrict ou
> country has about 14 million inhabitants, versus the
> hundred times as much. The area of The Netherlan
> versus the 33,000 square kilometres of the US, that
> weighed against each other yields for The Netherla
> fifth of that of the US.

In a test 312 future primary school teachers were as
paper cutting. The scores were both revealing and

Correct : 18
Wrong : 191
No answer : 103

Many students performed their calculations exactly by means of column procedures. This indeed was the most essential shortcoming which could be observed, because this resulted in the production of failures which were not in the article. Other mistakes were sloppy arithmetic and the wrong processing of magnitudes and big numbers (Jacobs,1986).

Remarks:
Obviously the future teachers (as well as some of the pupils of the former example) lacked the notion that and how numerical data are anchored in reality and, with regard to measuring, did not have to their disposal reference points such as the size of a football-ground, the size of a country, the number of inhabitants. Mathematics education should aim at developing personal scales of familiar and lived through measures such as:

- the distance between home and school, also measured in time, walking and biking;
- one's own weight and stature;
- one's walking and biking distance per hour;
- the height of a house, a twenty stores building;
- the size of the playground, a football-ground, and so on.

Such personal scales, the richer the better, form reference frames for solving problems of the kind as presented.

What do the foregoing examples mean in the context of realistic mathematics education? Of course they have a merit of their own but in the present context they have been adduced because of their constructive and productive value.
Educated estimates and implicit experiential data made explicit, strengthen the grasp on problems, which is one of the functions of construction and production. Solving means tying connections between the real and the arithmetical world by means of mathematical modelling.
Growing such connections helps developing a mathematical attitude, in particular horizontal mathematising, that is mathematising real world situations (cf.Treffers,1987). Almost all of the 312 future teachers lacked that mathematical attitude required to clean the mess of data in the newspaper cutting.
This proves that the environment where they learned mathematics differed much from that of the school lesson.
The spirit of the lesson is comparable with the direction in which the problem solving courses of Schoenfeld (1987, p.213) have been developed:

> 'With hindsight, I realize that what I succeeded in doing in the most recent versions of my problem solving course was to create a microcosm of mathematical culture. Mathematics was the medium of exchange. We talked about mathematics, explained it to each other, shared the false starts, enjoyed the interaction of personalities. In short, we became mathematical people.'

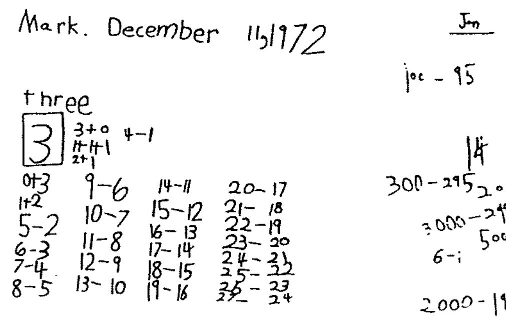
3.3.3 Seizing the opportunities of continued organi
ematical material.

Pupil's own constructions and productions mirror t
both for the teacher and the educational developer
examples.

The first

Grossman (1975) reports about unexpected surprise
She presents a few examples of work with first gr
and include the teachers' comments (ibid. p.14-15)

> 'Mark was having trouble with arithmetic until I g
> me and he proved to himself that not only he could
> stop doing it. (He handed in two extra papers on hi
> other children loved the activity too. My feeling wa
> they could do it all.'



> 'I knew Jon was bright because he understood so w
> tured lessons, whether I followed the syllabus or we
> er, I never suspected that he could handle numbe
> thousands. There I was, teaching combinations up
> tions and the children's ceilings.'

Remarks:

The teachers' comments show that both boys had
of the scholastic domain. Mark's work still reveal
flected on his activities. After a hesitating start w
arithmetic he screwed up courage, became self-c
sailed a fixed course through the system he built wh
He transgressed the boundaries of the arithmetic
structure. At home he continued intensively – the s
to have problems with arithmetic.

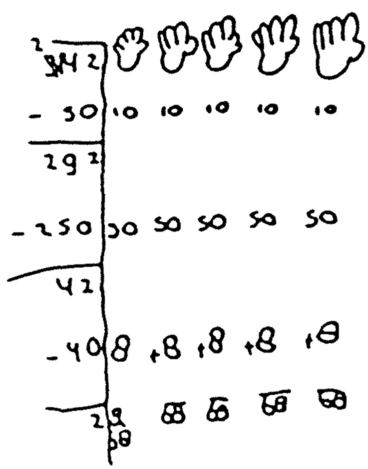
And then Jon! How much curtailed must he have been in his possibilities! He anticipated on sums, three grades higher in the curriculum; up to 10,000-9,995 = 5! Like Mark he worked systematically. Only his written report was a bit untidy.

Both of the boys reflected on what they had learned within the number system, and consequently they anticipated on the future of the teaching/learning process, the one farther than the other. The teachers were hold up the mirror of their instruction. Especially Mrs.S. (Jon's teacher) was conscious of this fact.
What would pupils' own constructions and production have mirrored in rich contexts of realistic instruction? The answer to this question can be found in numbers of publications (cf. Van den Brink, 1987).

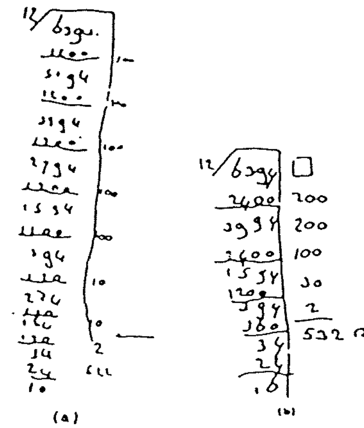
Second
A course of long division can be based on the principles of clever computation and estimating (cf. Treffers, 1987). Reference should also be made here to Gravemeijers contribution about contexts and the examples he gave. So I will confine myself here to a brief impression concerning the development of an algorithm for long division. Let the start be

> *'342 stickers are fairly distributed among five children; how many does each of them get?'*

In such a situation distributing shall be organised. First the stickers are handed out piecewise, but soon bigger shares are dispensed. The written report reflects the distributing pattern, which indicates the distribution process. Subsequent steps on the path of mathematising are predesigned.



In the second phase the children are soon satisfied w
umn only – 'all get the same, indeed'. Other co
among which that of grouping. After about 15 less
ferent levels.



In the third phase the connection is made to decim
according to powers of 1/10 becomes central but th
essentially. Context dependent answers on division
glected for instance interpreting an outcome with
showed already.
At crucial points in the course it is asked to invent
by a slow long-winded manner as well as by a qui
should learn to reflect on their learning process and
procedures.

Remarks
With regard to contents the course of long division
acterised as follows:

- a process of mental computation and estimatin lems;
- a process of progressive mathematising arithme case by means of schematising and shortening.

Such an approach of division starts with the infor
which are organised and structured. Construction a
tant part in the process of progressive schematising
pects of progressive mathematising. During the tea

lutions of applied problems are continuously subjected to inventarisation. Continuously the question of possible shortening is raised. The procedures arising in the course of shortening function in the course to be followed: beacons for those who nearly reached the same level of mathematising. The ultimate standard algorithm of long division is predesigned in this process as the utterly shortened procedure.
In a sense this mirrors the historical process of algorithmising long division (as well as the other operations on whole numbers; cf. Menninger, 1958). In fact the present course was at least practically inspired by the view on the historical development.
Comparative research undertaken in our country has proved that this approach is by far superior to the traditional one. An experimental group attained in half the time a result almost twice as good as a control group which had been taught traditionally (cf. Rengering, 1983; Treffers, 1987).

	exp.	contr.
difficult divisions (zeros in dividend or divisor, etc.)	85%	45%
applications	70%	45%

Scores in long division – traditional method vs. progressive schematising.

The results on the traditional method have been confirmed by other research, also in other countries.

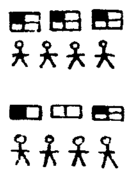
3.3.4 Uncovering learning processes and reversing wrong trends

We have already noticed the diagnostic value of own constructions and productions, mirrors as it were, illustrating the teaching as well as the learning process. At present we will consider the diagnostic value for the learning process, in particular cases where constructions and productions reveal wrong ideas and misconceptions.

Example

The class had elaborated and described two distribution situations (cp. Streefland, 1984; 1987). It was quite early in the teaching/learning process, after a number of suchlike activities in the past. The teacher judged it the just moment to proceed to the first task of free production in this domain. The pupils were challenged to think out such 'number sentences' as had been met in the distribution situations, that is, with halves, fourths and – for the courageous ones – eighths, with 'plus' and 'minus', maybe even with 'times', sums matching distributions.

Share three chocolate bars among four children:



Distribution performed and descr

Each will get. $\frac{1}{4}$ (of a bar) + -

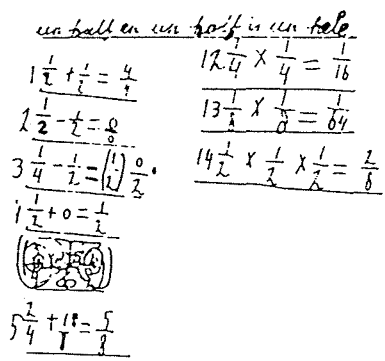
$3 \times \frac{1}{4}$

$\frac{3}{4}$, or

$\frac{1}{2} + \frac{1}{4}$ (= $\frac{3}{4}$) ea

Example of a distribution situati

Michael produced the following:



Michael's work is typical for - world-wide-mistake
(cf. Hart, 1981; Hasemann,1987; Streefland, 1984; a

Remarks

The diagnosis is clear.The mistakes are the conseque tation of whole numbers and their rules. It shows that tal object 'fraction' has not progressed far enough t merators and denominators were still operated upon interdependence was neglected.

The task had been set too early, at least for Michae been switched off prematurely. Stating this goes to t envisaged. In their own constructions and productio wrong ideas and misconceptions. In other words: Ow tions unveil the – possibly wrong– personal theoreti ticipation in the teaching/learning process. This enha

the material. A correct diagnosis promises successful remediation both of learning and teaching.
As a matter of fact this is closely related to ideas in Sinclair's and Vergnaud's PME-XI addresses at Montreal. Indeed, what has happened? Michael reckoned among others 1/4+1/4=2/8.
Remediation can start with eliciting a conflict. In the concrete (imaginable, meaningful) environment a new solution can be tried: Four children share two pizzas.Make a distribution. How much does each of them get? The solution may be  (one fourth and one fourth are two fourths, which equals one half). There are children who do not experience this as a conflict (cf. Streefland, 1984; Hasemann, 1987). In these children's conception the result (still) depends on the solving method, that is, on the level at which the solution is conceived (concrete vs. symbolic). This level dependence is an example of what Sinclair (1987) named a 'normative fact', and Vergnaud (1987) 'theory in action'.

3.3.5 Producing terminology, symbols, notations, schemes, and models serving both the horizontal and vertical mathematisation

Children learning mathematics can, by their constructions and productions, contribute to its working apparatus.

Example 1

Madell (1985) reported about the personal algorithms for subtraction, developed by pupils of the Village Community School in New York. Their 'natural', informal methods of performing the operation had the following characteristics:
- both working (partly) per column and from left to right;
- working with position-values in stead of the numbers per position;
- working with deficits and borrowing from tens; no child applied the standard procedure of borrowing;
- working along the lines of proceeding abbreviation.

Let us have a look at an example, reflecting some of the features just mentioned.

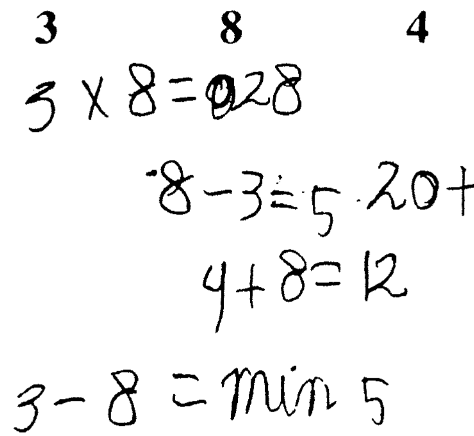
Stephen's Running Total (Abbreviated)

```
  8371        8000 - 3000 = 5000
- 3754         700 - 300  = 400
              5000 - 400  = 4600
                70 - 50   = 20
              4600 + 20   = 4620
                 4 - 1    = 3
              4620 - 3    = 4617
```

How can you explain the combined usage of addition and subtraction in Stephen's method?

(quoted from Labinowicz,1987, p.381).

This own invention of column subtraction can be us
standard algorithm, deviating from the usual one as h
one.
Even young children entering the set of negative num
example illustrates the awareness of an eight years ol
'minus' representing the operation subtraction and 'r
tive state of the outcome.



Remarks
The self-constructed notation served the developme
umn subtraction. While working from the left to the
minus 3,000 equals 5,000, 300 minus 700 is 400 short
ground reasoning that subtracting 300 is still possible
tracting 400 more brings ⁻4 at the place; and so on.)
The deficits can also be indicated by upper dashes,
pupils work).
For the transition from 5423 to 4617 money can be
tional material: property 5,000 debt 400........This is a
informal methods to start a process of algorithmising,
tion and steered by progressive mathematising.
Moreover the pupils do not loose sight of the glob
volved.The production of new forms of notations m
course of thought and creates the possibility to short
Regularly such approaches are met with in publicatio
sible consequences for the outline of the programme
are not recognized, dealt with and elaborated. Anyho

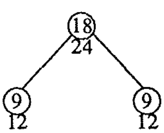
ing to what has been shown with respect to this, the usual algorithm for subtraction would have to leave the field (after Treffers, Feijs en De Moor, 1988).

Example 2

Dividing per unit and several units simultaneously in distribution situations is an opportunity for children learning fractions, to produce equivalencies by themselves. In the distribution activity 1/4 and 1/4 go together with 1/2, and 1/2 can be decomposed (among others) into 1/4 and 1/4. During our education developmental research (cf. 4) pupils contrived such terms as 'hiding name' or 'conceal name' to indicate non-standard names for fractions. Such terms facilitated the communication but also described it efficiently. (The Dutch word 'schuilnaam' sounds less 'learned' than 'pseudonym'.) The quest for a fitting term for some (mathematical) phenomenon can elicit reflection, as this example shows.
The most suitable propositions that were offered, also proved to have a long term predictive value (cf.Treffers & Goffree, 1985; Streefland, 1988).

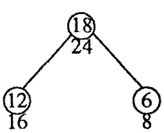
Example 3

In more extensive situations such as 'dividing 18 pizzas among 24 children' the actual distribution, whether pictorial or imagined, is too laborious. In our developmental research some children got to use the service at tables as a means to reduce the situation to manageable proportions.Thinking about it they found out the symbol $\frac{18}{24}$ for 24 children around a table with 18 pizzas. This made it possible to represent the service at tables on paper. It led to organising and structuring activities such as building schemes that expressed variations in table services. For instance:



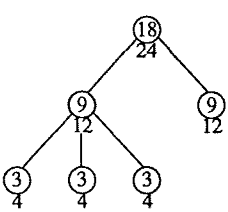
that is, two tables $\frac{9}{12}$ instead of one $\frac{18}{24}$

or:



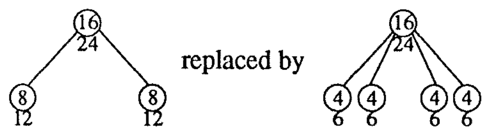
with the tables $\frac{12}{16}$ and $\frac{6}{8}$

This can be continued:



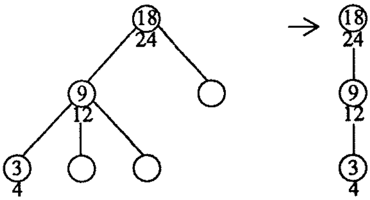
At each given moment this schemes-building can b
everybody's fair share can be determined by means
distribution situation is being made accessible to the
of fair sharing at one table and the fair table service.
will reconstruct the food/consumers relation. This l
cept of fraction and ratio in their mutual relation. Th
organises in an almost evident way the production o
gard to fairness. The food/consumers relation of th
mentally continued at each step: *ratio conservation.*
table service in the background. The context situati
the model situation of table service becomes a situ
Goffree, 1985; Streefland, 1986).

Remarks
The quest for fitting symbols for distribution situatio
table services supported by this symbol elicit reflecti
of horizontal mathematisation.
Distribution situations are being located within math
ing encouraged by the opportunities of progress
emerge as naturally as in the example of long divisi
The self-contrived symbol and the patterns in which
situations with each other by decomposing them i
which can more easily be compared, for instance tabl
guests.
The symbol $\overset{\textcircled{3}}{4}$ is a metonym for the situation and th
uation model of table service, which functions as a c
Greeno, 1976).
Continuously applying the scheme leads to two typ
cover the reflection on the own activities.
The first is scheme-conserving while the notatio
branches, or at least the numbers are omitted so tha
service are respected. The second is shortening in de

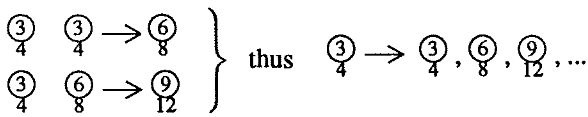


which changes the pattern of the table service with t
their common divisors steering the shortening.
This involves level-raising in the learning process. Th
el is attained when the pupils consciously and system
tion by means of the greatest common divisor while
this idea.

Moreover this may lead the learner to focus on proportion tables:



Starting from a given portion 3/4 and looking for matching table services connects 3/4 and $\underset{4}{\textcircled{3}}$ with each other; while both of them are still distinguished by the difference in notation. Pushing tables side by side generates new tables granting everybody the same portion:



The next step indeed is ratio tables. (Streefland, 1985).

Example 4

Nineteen years old Fynn (1976) told a fascinating story on six years old poor Anna. She had her own way to manage big numbers. She knew that big numbers could be made ever bigger but she lacked words to express them.

When she would transgress the limits of millions and billions she, in order to continue, invented squillions.

Some fine day she told Fynn she could answer a squillion questions. 'Me too', Fynn said unimpressed, 'but, among half of them wrong'. 'Not so,' Anna said, 'all will be good'. 'Idle nuts', Fynn thought, 'nobody can and she the least'. She deserves rebuke. But Anna did not take it.

'How much is one plus one plus one?'

'Three, of course'.

'How much is one plus two?'

'Three.'

'And eight minus five?'

'Also three.'

Fynn wondered what she was getting at.

'How much is eight minus six plus one?'

'Three.'

'How much is one hundred and three minus one hundred?'

Fynn interrupted; he felt she was pulling his leg. She was inventing the problems on the spot and could go on that way until the cows came home. Nevertheless Anna enthusiastically made her last move.

'How much is one half plus one half plus........'

Fynn had got the message.

'How many problems can be answered by three?'
'Squillions,' Fynn said.
'Isn't it funny, Fynn, every number is the answer to
(paraphrasing from the Dutch version)

Remarks
This own production reveals high level reflection, ty
titude in the spirit of Krutetskii (1976) and Freuden
left to the reader. In the next section this example wi
The foregoing examples show the part played by the
symbols, notations, schemes and models in the shapi
izontal as well as vertical.

3.4 Education developmental research for the sak struction

Up to now the stress has been on (re)constructive le
er's perspective. It started with informal notions and
struction gradually moved the learner towards mor
tions, operations and structures.
(One of the examples was concerned with *unlearnin*
notice that often too little attention is paid to the pote
in instruction (cf. Freudenthal, 1983).)
The import of reconstructive learning is also at the he
riculum developers and researchers are seldom aw
than seriously observing children and learning fron
structions and productions, they expect answers on
problems by prematurely theorizing within topical fr
sounded time and again in the literature on developm
tened to. The results of didactical research in teachi
glected. Fractions is a telling example: fresh starts
Nothing is learned from lessons such as taught by
like Freudenthal (1968)(1973), Hilton (1983) or Usi
A striking illustration of this fact is Brownell & Cha
the results of drill for the mastery of basic skills con

> '..... the time and accuracy scores on Test B were bet
> the month's drill had materially raised the level of th
> cause drill had supplied more mature methods of thi
> because the old methods were employed with greater

By 'old methods' the authors mean pupils' own info
instruction against the grain. Wouldn't we have ma
knowledge about childrens' mathematical learning i
ing results of research?
An important question now is: Does reconstructive l

dinally to class instruction? Our reports related to the seize of The Netherlands, long division and table service provide indications for group learning processes. In order to answer in the affirmative, we have to carry on developmental research– which means research in action.
It aims at developing prototypes of courses and theory-building for teaching and learning in a certain subject area. Instruction experiments start with provisional material. The teaching/learning process is closely observed.

Continual observation and registration of individual learning processes is at the heart of the research. What matters is that pupils' constructions and free productions are *used for building and shaping the teaching course.*
In the variety of children's possible proposals (look for the kind of problems to be used (2)) one gets a rich choice to find out what is the best fitting, the farthest prospective, and in the long run the most effective. Blocking and diverting material is eliminated.
This is no illusion. At ours as well as abroad courses have been developed in this way (for science, see Driver (1987); for mathematics, see Treffers (1987)).
With the aid of children's constructions and productions even a course for fractions closely tied to ratio and proportion has been developed (Streefland, 1988).
In this kind of design children, by their learning processes, decisively influence course development – this even extends to supposedly weak learners as some examples proved.
It nourishes the source for creating reconstructible instruction. The prototype can serve as a model for establishing and developing derived courses. Such potential instruction is predesigned in textbooks and manuals. Globally the used generative problems with pupils' usable long-term constructions and productions, which emerged in the developmental research, will mark the learning road for fresh pupils' cohorts. In particular the manual will prefigure the material to be expected from the pupils and help to reorganise it with the view on the sequel. This is a means to realise teachers aided reconstructible instruction. Or, to express it differently: In this way the preconditions are fulfilled to have the teachers treated their instruction as free production of teaching; that is teaching also brought forth on the base of the constructions and productions of the pupils.
Such teaching, rather than transfer of knowledge, is negotiation of meanings (Driver, 1987, p.8). No longer does the course represent the teaching contents but:

> '..... a programme of learning tasks, materials, and resources which enable students to reconstruct their models of the world to be closer to those of school' (mathematics; added by L.Streefland) l.c. p.8).

3.5 Conclusion
The construction principle in education requires a significant part played by children's constructions. What this means for mathematics education has been

showed earlier in the introductory chapter on real
and its theoretical framework.
Within realistic mathematics education a solid emp
ciple of constructivity by having the children contri

Horizontal and vertical mathematisation as observ
process can be a source of inspiration. In the light of
ing is realised on the individual as well as on the cl

Notes

1. Revision of Streefland (1988).

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4 Realistic Arithmetic/Mathematics Instruction and Tests

Marja van den Heuvel-Panhuizen

4.1 Introduction

According to the different ways of teaching there are different ways to find out what children have learned by the given instruction. In fact each didactics has its own means of evaluation, chosen among those that fit the given instruction the best. So quite often the means used in testing reflect the characteristics of the instruction.

For instance, in the case of realistic arithmetic/mathematics instruction – that is how we call it in my country – one cannot possibly limit oneself to multiple choice tests. Indeed, they don't do justice to the goals of this kind of instruction, and they fail completely as soon as the children's developments are to be tracked. Realistic arithmetic/mathematics instruction asks for other means of evaluation. Instruments that are preeminently suited for this instructional approach are observations and interviews with students.

Observations and interviews create the possibility to uncover the children's strategies, to have the children reflect on the strategies they applied, and – by adapting the questions to what the children are doing and telling – to trace the children's knowledge and abilities. Written tests don't look appropriate for realistic arithmetic/mathematics instruction, where this aim cannot possibly be attained with the usual ones.

4.2 Drawbacks of tests

Let me show you an example!

1	2	3	4
1 + 6 = ...	8 - 4 = ...	7 + 2 = ...	8 - .... = 7
2 + 7 = ...	9 - 3 = ...	.... + 3 = 5	6 - 5 = ...
6 + 3 = ...	4 - 0 = ...	4 + ... = 6	.... - 4 = 3
8 + 0 = ...	10 - 6 = ...	5 + ... = 8	.... - 9 = 0
6 + 2 = ...	5 - 5 = ...	9 + 1 = ...	10 - .... = 9
3 + 5 = ...	7 - 6 = ...	.... + 8 = 9	7 - 3 = ...
4 + 4 = ...	3 - 2 = ...	3 + ... = 7	3 - .... = 2
0 + 9 = ...	6 - 3 = ...	.... + 4 = 9	.... - 2 = 3
5 + 1 = ...	2 - 1 = ...	.... + 5 = 6	.... - 5 = 3
7 + 3 = ...	10 - 9 = ...	6 + ... = 10	9 - .... = 5

Testblad 1 3

fig. 1

This is one sheet of the 'Schiedam Arithmetic Test'[1], used in the Netherlands. The children are allowed a restricted time to complete the test. Each series represents a certain type of sums of increasing difficulty: numbers grow larger and larger and operations become more difficult. First adding up to 10, then subtracting under 10. Subsequently indirect additions and subtractions, then arithmetic up to 20, which in turn is followed by additions and subtractions up to 100.
This kind of tests suffers from two huge drawbacks.

1. The first drawback is that the tests reveal only the bare results and tell nothing about the children's strategies. This lack of information on children's strategies has the consequences that:
   - wrong conclusions are likely to be drawn on the children's performance; indeed good answers can have been got by mere chance;
   - too little information can be obtained about the progress of instruction; for instance, nothing is learned about he students' informal knowledge and solving methods;
   - and finally it is almost impossible to diagnose the children's arithmetic/mathematical problems; any error analysis that solely depends on the results can never suffice to discover the children's problems and misconceptions.
2. The second drawback of these tests is that they are too narrow, both with regard to the subject matter and the students:
   - restricted to such subject matter as can easily be tested;
   - not allowing the children to optimal show what they are able to; perhaps lacking abilities are balanced by others that don't get any chance.

Yet on the other hand written tests have the big advantage that in one trial and in a short time a whole class can be examined as to whether certain abilities are mastered or not. This advantage implies that even in realistic didactics written tests cannot be brushed aside as long as no suitable alternatives have been searched for.

4.3 Alternatives

With regard to secondary education in the Netherlands, the awareness of the drawbacks of the usual written tests from the viewpoint of didactical change and of new curricula has meanwhile led to shaping new testing instruments. In the HEWET-project new test instruments have been developed at equal pace with the new curricula in order to do as much justice as possible to the principles of realistic instruction. The subject of tests is a momentous part of Jan de Lange Jzn's thesis 'Mathematics, Insight and Meaning'.[2] Among suitable written alternatives for traditional tests De Lange investigated what he named 'the two-stage task', 'the take-home task', and the 'essay task'. Since it would take us too far dealing with these test instruments, I shall restrict myself to repeat the five starting points for developing the new testing instruments:

- the tests are to contribute to the learning process and to further the students' progress, rather than being considered as a final piece;
- the tests are to enable the students to demonstrate their knowledge and abilities rather than their deficiencies (positive testing);
- the test contents must cover the instructional goals as much as possible; they should not be restricted to measuring the knowledge results; processes, that is, the way students reached certain solutions, are more important than products, that is, the eventual results;
- rather than the feasibility of objective scoring, the test contents must be the primary quality criterion; objectivity is a secondary aim;
- the testing instruments must be easily applicable in the classroom situation (cf. De Lange, 1987, p.179-181).

4.4 MORE-project

The previous starting-points together with the principles of which have been expounded in the foregoing contributions of my colleagues, have played an important part in the search of the MORE-project[3], for better instruments in primary education.

The project is an investigation into the implementation and the effects of realistic didactics if compared with the traditional mechanistic brand. To answer the question of the effects, among other data, students' learning results had to be collected, and since their number amounted to about 400, written tests was the only feasible procedure. (Aside it may be mentioned that a small part of the students population has individually been interviewed.) The very problem of the project was the unavailability of tests, except for the usual written ones with the earlier mentioned drawbacks, consisting of bare sums. This meant that new tests had to be developed, which avoided these drawbacks as much as possible. In short, we needed tests:

- that covered the whole spectrum of the arithmetic/mathematics area concerned;
- that gave children the opportunity to show what they are able to;
- that provided information about abilities and strategies;
- that could easily be handled in a classroom situation.

I am going to illustrate these points more in detail and to explain by means of examples how they were made concrete in test items. Although the various items diverge in the measure of fulfilment of our four requirements, it has been tried to have each item to satisfy all of them as much as possible. The examples to be shown apply to grades 1 to 2.

4.5 Easily to be handled in a classroom situation

I take the last point first: the tests shall be easily administered in the classroom. This is no problem for written tests, consisting solely of bare sums. After the test booklets have been handed out, the children immediately know what they are ex-

pected to do. This holds to a lesser degree as the children are given other kinds of problems but the well-known ones. New problems usually require some explanation, which makes administering in the classroom less easy. In the MORE research, however, we have tried to contrive tasks that can easily be given in the classroom situation with *a minimum of explanation* – anyway, no tasks requiring extensive oral or written instructions, which would result in a reading or listening comprehension rather than in an arithmetic test. Textbook-specific jargon and procedures had to be excluded as well. Instead we have looked for tasks, accessible to each child, self-explanatory tasks, requiring no additional information beyond that minimum of instruction that is needed to get the intention across. As examples take the following items.
The first item (fig.2) is related to a game of darts being thrown at a target. The question asked is: 'How many points together?' Although the picture is not quite faithful, together with the question it is sufficient. The children immediately grasp the intention. The same holds for the next test item (fig.3). The question is simply: 'How many florins do you keep?' Each child understands that a toy train is being bought, and the picture shows its price and the money in the purse.

Test 4.2 (8)

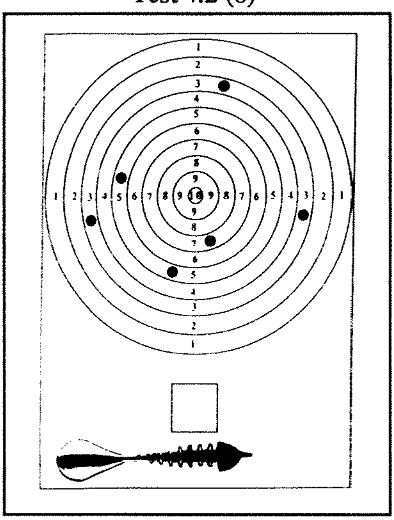


fig. 2

Test 3.4 (16)

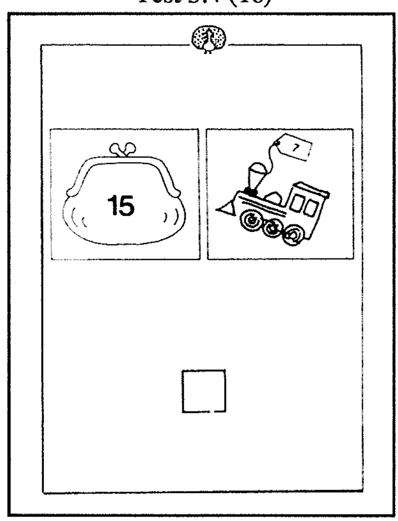


fig. 3

One more example (fig.4): 'Though you cannot see all cans, can you tell how many there are?' – a simple question, though not necessarily a simple task, which requires more than mere counting the visible cans, indeed.

Test 3.2 (4)

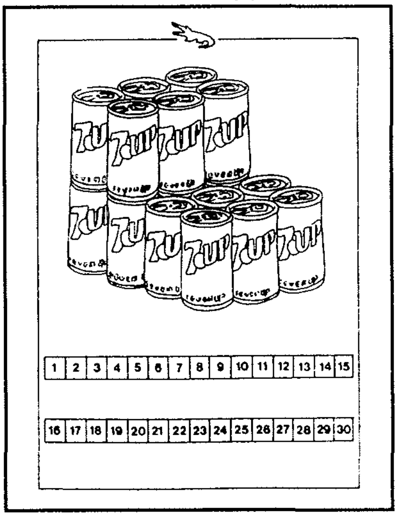


fig. 4

4.6 Covering the whole spectrum of the arithmetic/mathematics area concerned

Though the point of covering the whole spectrum of the arithmetic/mathematics area has been anticipated, as it were, by the last item, it will be accounted for by more examples.

Test 3.4 (21)

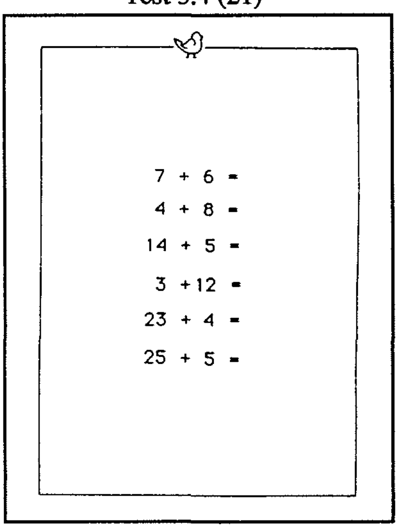


fig. 5

Test 4.2 (2)

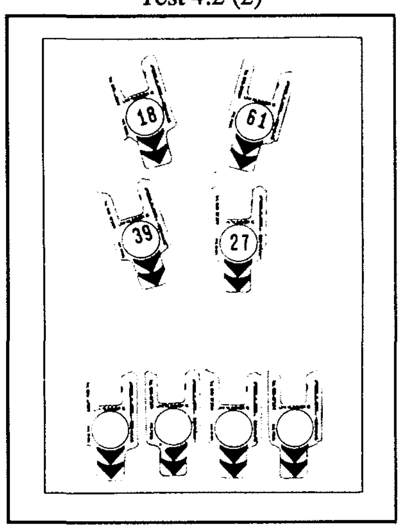


fig. 6

Test 4.1 (11)

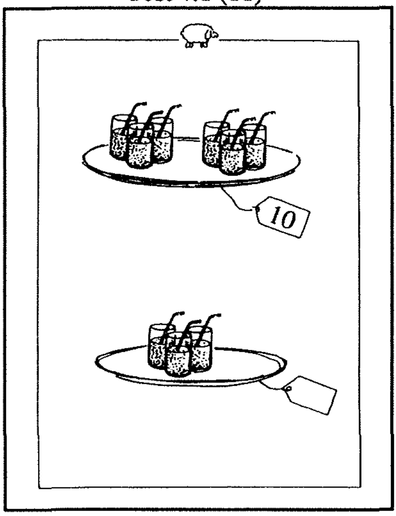


fig. 7

Test 3.4 (20)

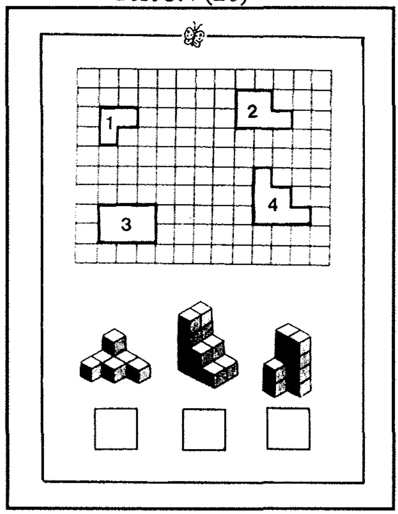


fig. 8

It means that *besides* the usual series of *sums* (fig.5) *other chapters* of the arithmetic/mathematics area concerned have to be represented, such as the chapters: counting and counting sequence, ratio, geometry. An example of the first is shown by fig.6, where the children are asked to order waiting numbers, as are often used in offices and shops, according to their magnitude, from the left to the right. The next item (fig.7) has to do with ratio. The question is: "How many florins for three glasses of juice?" And in the geometry item (fig.8) the children must find out the place where the block buildings may have stood on the grid.

Covering the whole spectrum does not mean, however, that each chapter is tested separately. On the contrary, it is even more important that this happens in their *mutual connection.* As an example take the item with the fishes on the roller blind (fig.9). It looks a bit like the one with the cans, which was shown earlier. The children are asked whether they can know the number of fishes in spite of the obstruction of sight by the cats. Obviously this task asks for more than simple resultative counting. It is as well related to measuring, geometry and ratio.

Another example of simultaneously tested abilities is the next item, on cookies (fig.10). One of them costs 20 cents, and the question is the price of the others. Again a numerical operation, measuring, geometry and ratio play parts in this item.

Test 3.4 (2)

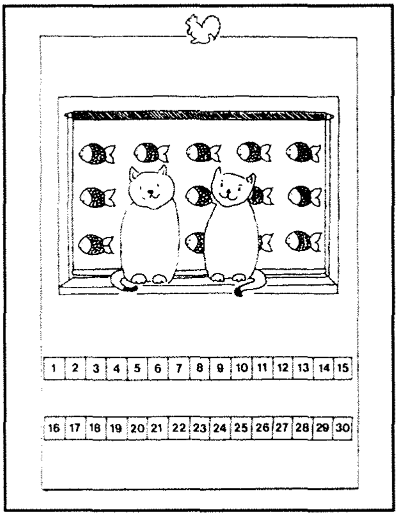


fig. 9

Test 3.4 (13)

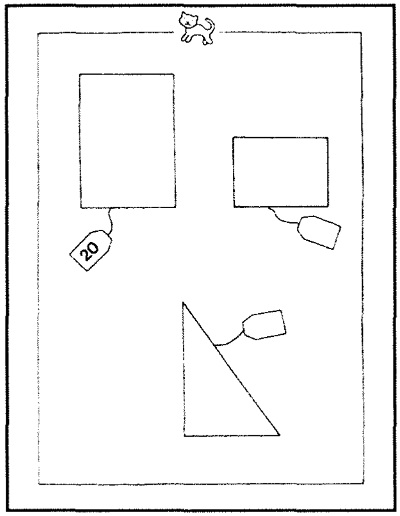


fig. 10

Besides covering all parts of the spectrum, separately as well as in connection with each other, it is required under this point that the testing instrument allows for various *levels of applying* mathematics in reality, which includes analysing the problem as well as selecting and, if need be, supplementing relevant data.

Tasks that don't require much problem analysis as their presentation includes the operation to be carried out as well as all relevant data, are low level under the viewpoint of applying mathematics. High level means large own contributions of the children to the problem analysis.

An example of low level is the item on the jogging suit (fig.11), where the price of the whole suit has to be calculated. Without much ado the picture and the question tell that the numbers must be added. The item (fig.12) on the birthday treat – a habit in my country – is quite different. The question runs: 'If there are 30 children in my class, how many bags shall I buy?' Obviously this is a higher level of application. First of all, the arithmetical operation is not straightforwardly given; moreover even calculations need not yield the adequate answer (such as '30 : 9 = 3 rem 3' in the present case). Applying means more than translating pictures into sums. This is patently obvious in the next item (fig.13). It aims at a bike trip from one place to another and back. On the way out they stop at this sign post. The question is how many kilometres have still to be cycled from this point onwards totally.

The remaining points (children's opportunity to show what they are able to, and information about children's abilities and strategies) are actually the hard core of the search for alternatives to the traditional written tests. Though I will deal with both of them separately, their close connection will soon become visible.

Test 4.4 (7)

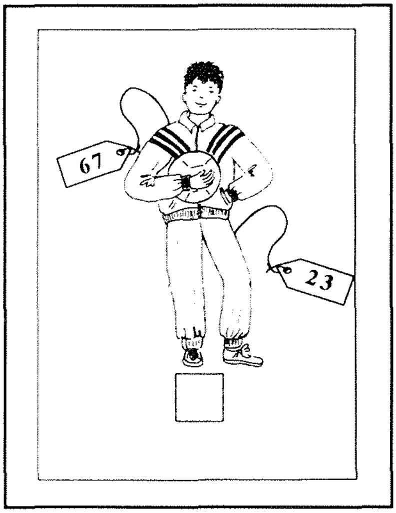


fig. 11

Test 4.2 (19)

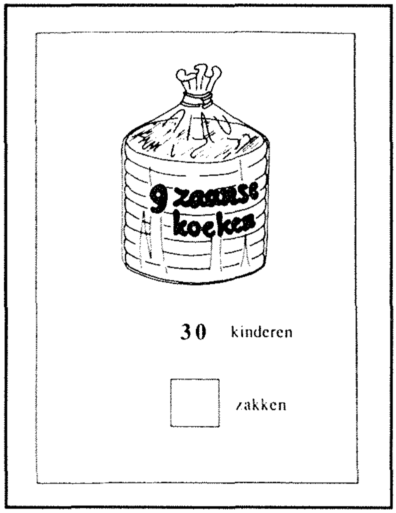


fig. 12

Test 4.4 (6)

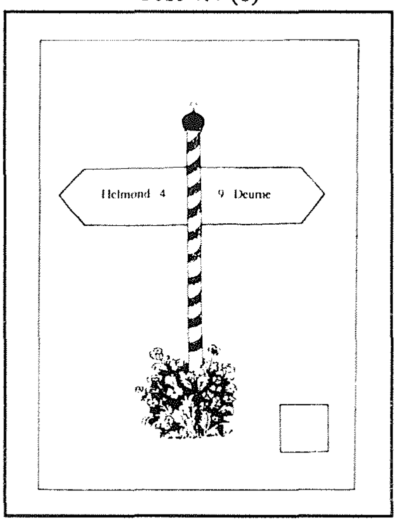


fig. 13

4.7 Giving children the opportunity to show what they are able to

The opportunity given to the children to show what they are able to should be given both ways, that is, both poor and bright students should profit.
Realistic didactics provides a number of useful clues, such as:

- using stimulating and supporting contexts;
- tasks with a number of built-in solving levels;
- possibilities for children's own contributions, for instance, own productions and choice tasks.

Stimulating *contexts* were offered in many of the earlier items. Besides those with a general stimulating effect one can offer contexts which both link up with informal experiences and function as supporting models. An example is the item (fig.14) where the knowledge of the counting sequence is tested with reference to house numbers (even or odd according to the sides of the street). The children are asked to complete the sequence.

Test 4.1 (5)

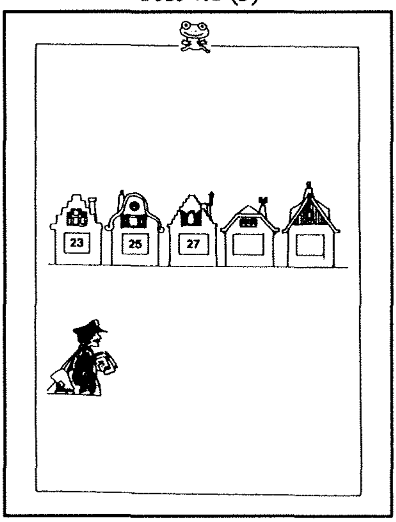


fig. 14

Another example is a buying situation (fig.15) where money functions as a model. Children at a level where 100 – 85 is still too difficult as a sum, can solve it via the change - a dime and a nickel. If the same test also contains the sum as such (fig.16) one can make sure how far children have progressed, that is, whether they still need the concrete orientation basis.

Another kind of elastic tests is that with *built-in solving levels*. This has been intended with the next (fig.17). A pair of dice has been cast and the question runs: 'Where shall the counter stand?' The children who don 't know yet that two and four equals six, can find the solution for instance by counting the points of the dice.

Test 4.4 (10)

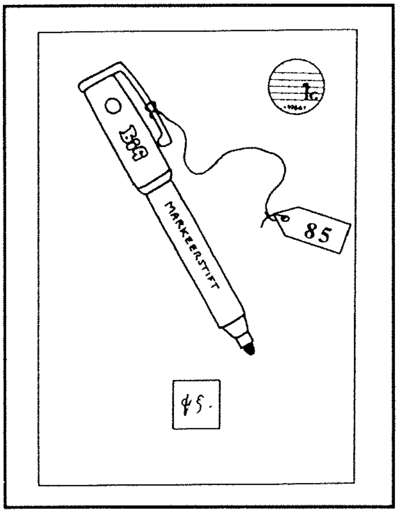


fig. 15

Test 4.4 (22)

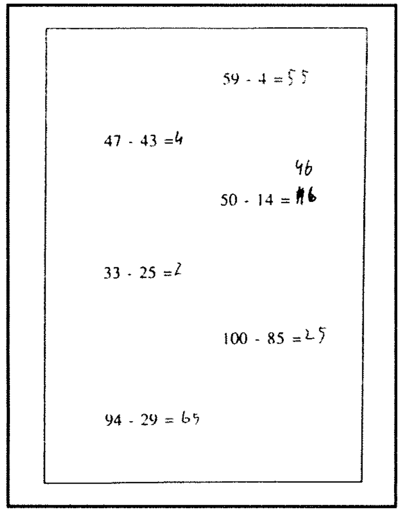


fig. 16

Test 3.1 (16)

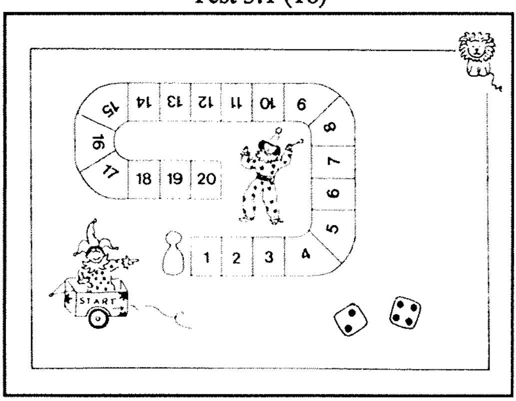


fig. 17

Another example of a layered test item is the one on 36 sweets which are fairly to be shared by three children (fig.18a). It requires a rather difficult division - difficult after three months in the second grade. Yet more than half of the children under investigation appeared to manage it, and the next sheet (fig.18b) tells why. It also shows that built-in solving levels is an excellent tool to get as much information as possible about applied strategies.

Test 4.2 (17)

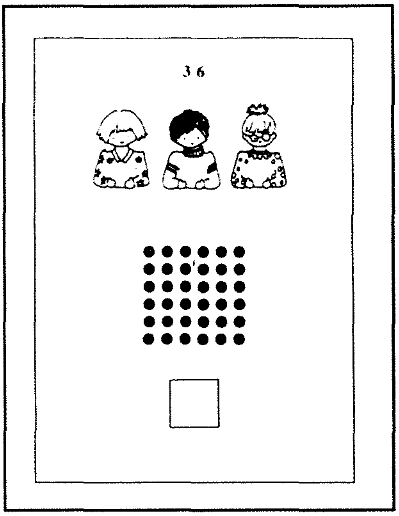


fig. 18a

Test 4.2 (17)

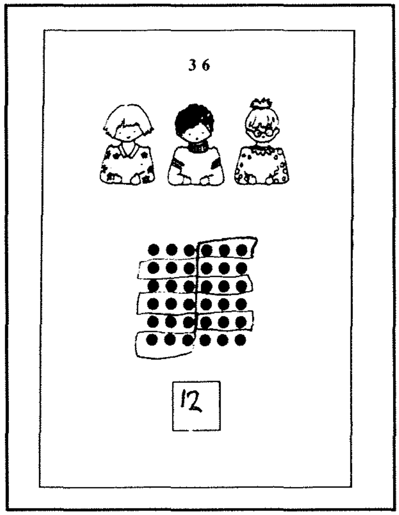


fig. 18b

Flexibility of design gives the children a lot of opportunities to show what they are able to, and the best way to realise it is creating possibilities for their own contributions. This can be done in various ways.
One way to do it is administering *test items with more than one correct answer*, which yields the children a latitude to contrive solutions. An example is the item (fig.19a) where in order to buy 12 candles the children may choose among the boxes as they like it provided it comes down on 12 candles together. Again it appears (fig.19b-c) that more latitude granted to the children makes the tests more and more informative.

Another way to grant latitude is *choice tasks* where the choices are built into the task itself. With the next two items (fig.20a-b) the children may choose for themselves what to buy. Besides a long sequence of degrees of difficulty, this choice creates indications on what the children are able to. Of course, preference for a certain object may have played a part, but it strikes that quite a few children make numerically similar choices in both items of a pair, as shown on the next sheets (fig.20c-d). One child chooses in both items an object that costs less than 5 florins, while another decides for more than 5 florins in both cases.

Test 3.3 (17)

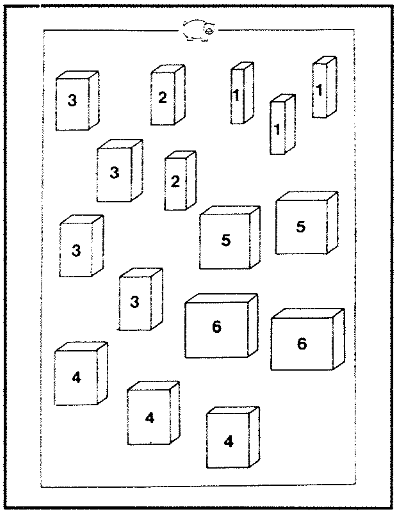


fig. 19a

Test 3.3 (17)

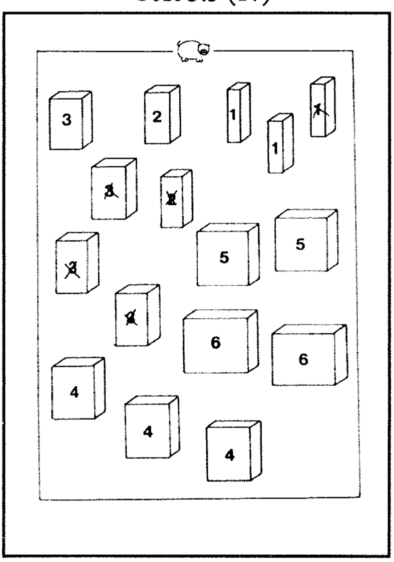


fig. 19b

Test 3.3 (17)

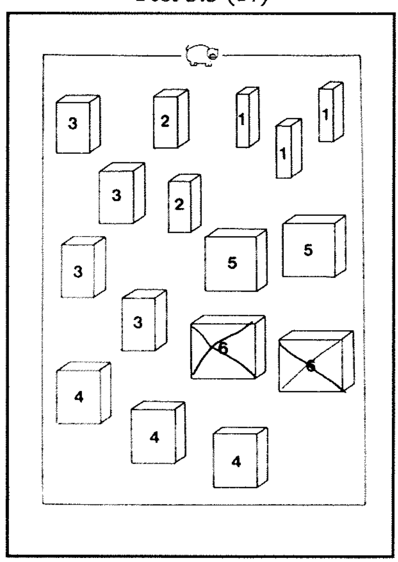


fig. 19c

It should be noticed that in the design of the test the number 10 had been omitted on the first of these test sheets (see fig.20a).

Test 3.2 (18)

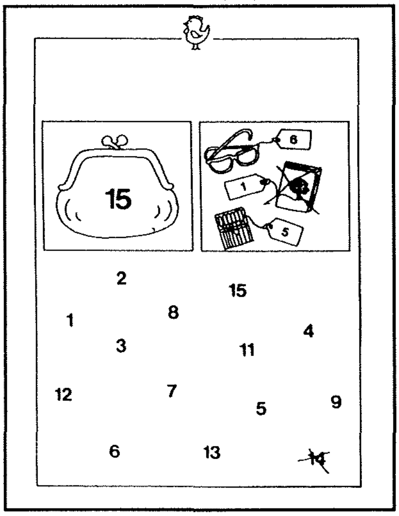


fig. 20a

Test 3.2 (19)

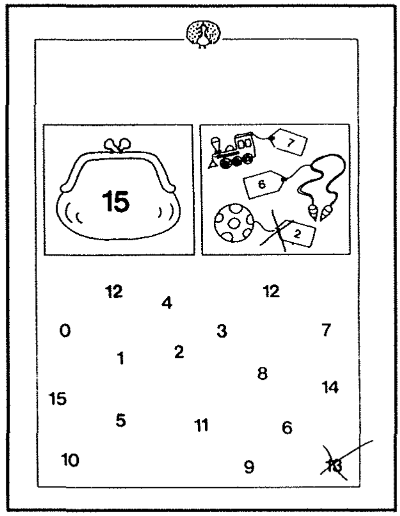


fig. 20b

Test 3.2 (18)

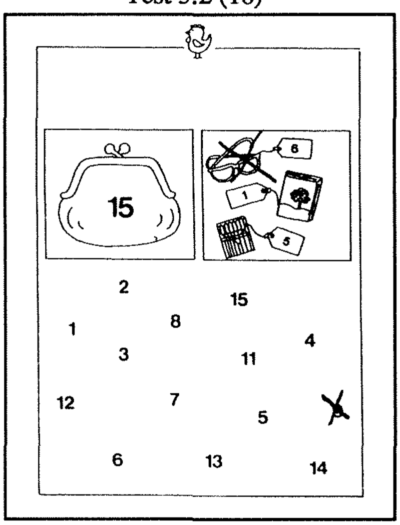


fig. 20c

Test 3.2 (19)

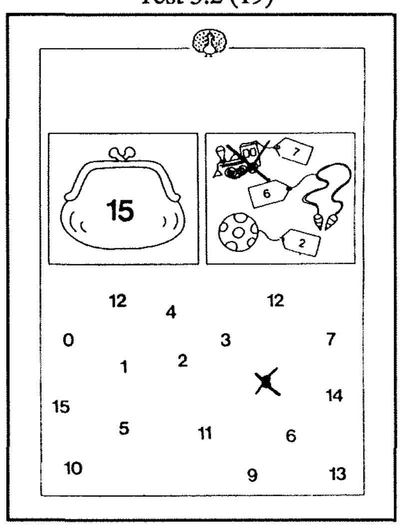


fig. 20d

Though unintended this mistake created one more opportunity to uncover the children's procedures.
The first child (fig.20e) adds himself the lacking 10, while the other (fig.20f) eventually passes to another object.

Test 3.2 (18)

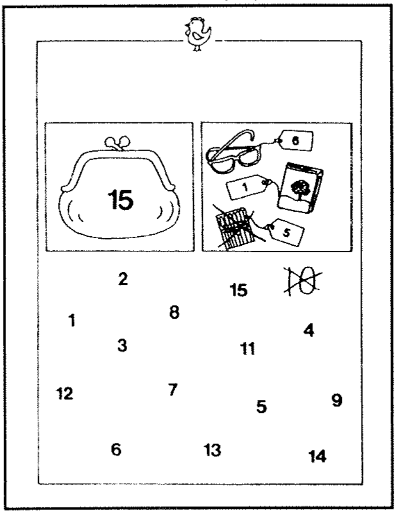


fig. 20e

Test 3.2 (19)

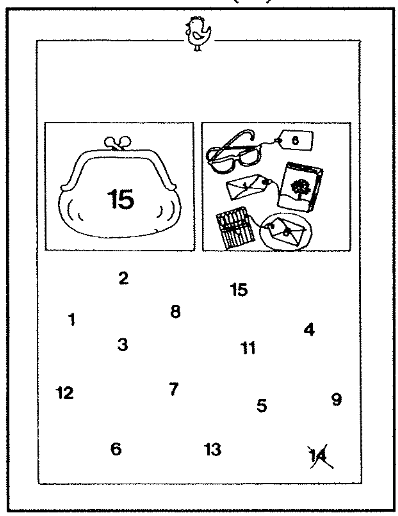


fig. 20f

The best way to have children showing what they are able to, is provoking their own productions. An example is the item (fig.21) where the children got the task to divide a cake into two equal parts in four different ways. The solutions given by this child prove that this task is well suited to have the children showing what they are able to do.
By the next item the children are asked to invent as much sums as possible with a given set of numbers. Let us look to one of the results! (See fig.22a.) As you see (fig.22b), at the end of the 4th grade this child even produces a sum with a resulting negative number.

A simpler problem is that where only one number is given and the children are asked to make as many sums as possible with this number as a result. Again this item is of the kind that both gives children the opportunity to show their abilities and provides information about the children's procedures. The one works systematically (fig.23a), while restricting itself to one kind of sums; the other contrives a great many different kinds (fig. 23b). These own productions can also reveal problems of the children which would never have been discovered in ordinary written work where the final result is the only information available. Look, for instance, for the wrong use of the equality sign, as occurring on the next sheet (fig.23c)!
Besides about the children the own productions yield information about the kind of instruction the children received, by reflecting it as it were. Besides this the own productions can excellently serve as instructional material.

Test 4.4 (20)

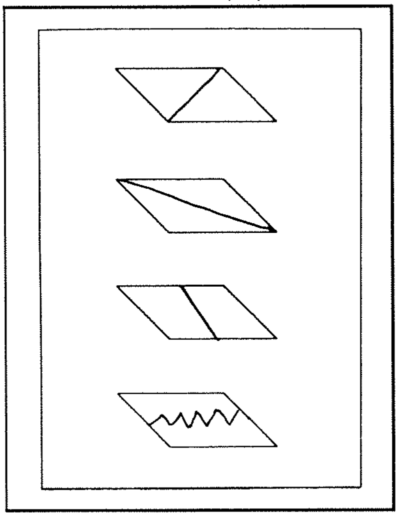


fig. 21

Test 4.4 (5)

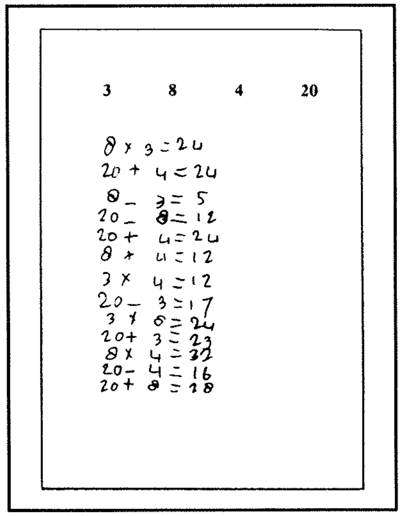


fig. 22a

Test 4.4 (5)

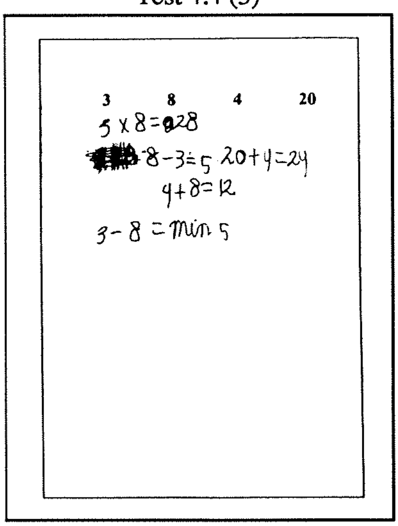


fig. 22b

Test 4.2 (20)

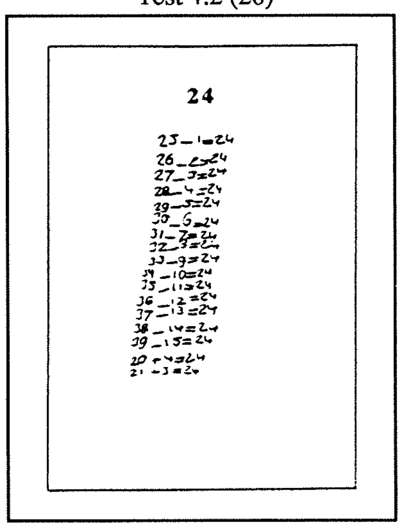


fig. 23a

A question that certainly arises after these reactions of the children is that of the consequences for scoring. Indeed, this *can* be a problem but it need *not* if one

clearly distinguishes between two kinds of information, provided by the test, quantitative or qualitative information.
The quantitative information is the number of correct answers, which can be used as an indicator for the level of performance of the subject.

Test 4.2 (20)

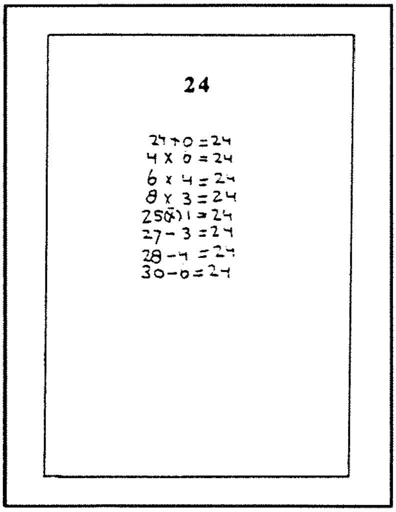


fig. 23b

Test 4.2 (20)

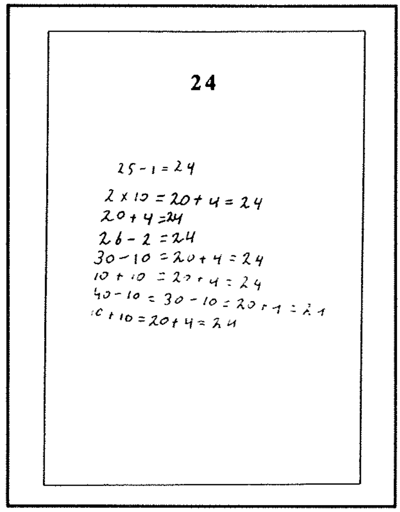


fig. 23c

This quantitative feature extends to the own productions provided a certain criterion has been agreed on in advance. Thus open tasks need not at all endanger the objectivity of the test.
Besides quantitatively the test informs qualitatively. This information don't lend itself for objective scoring. Often it cannot be settled whether a strategy is right or wrong. But this does not matter as these data have a quite different function.

4.8 Providing information about abilities and strategies

It has already been stated that the criterion of yielding information on abilities and strategies is closely connected to the preceding one. Much that contributes to the flexibility of the tests, provides a variety of information as well. We are going to repeat the resources available:

- tasks with various built-in solving levels (remember the fair sharing of the sweets);
- possibilities for children's own contributions, for instance, own productions and choice tasks (remember the earlier shown items where even sums must be invented).

Two more means may be added:

- presenting certain data in order to look what use the students make of them;
- stimulating reflection by the use of 'pieces of scrap paper'.

Test 3.4 (23)

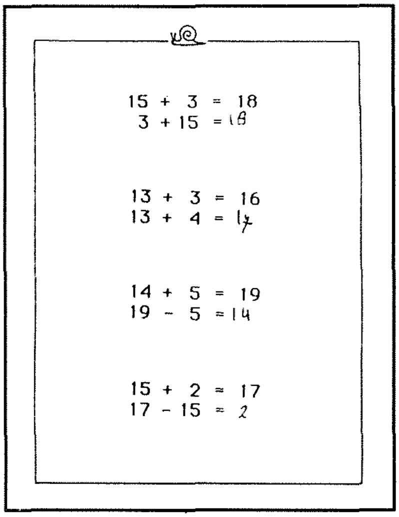


fig. 24a

Test 3.4 (22)

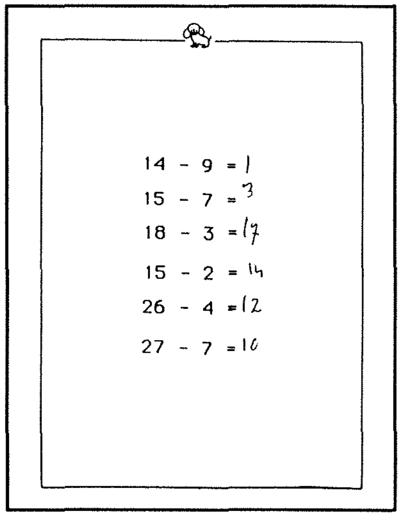


fig. 24b

Test 3.4 (22)

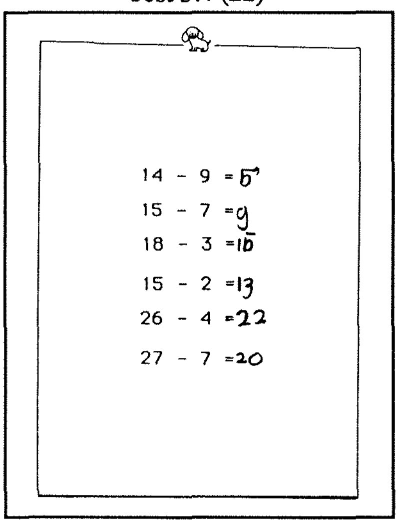


fig. 24c

Test 3.4 (23)

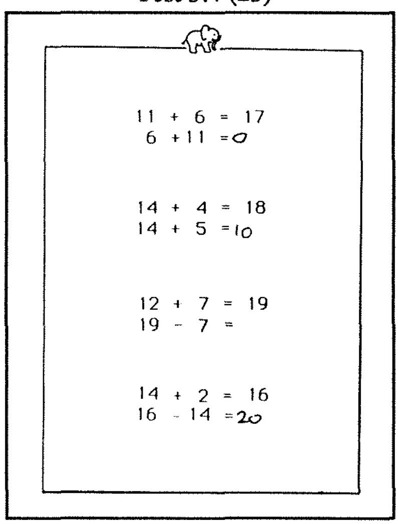


fig. 24d

The first is a means to purposefully look for the insights that children have acquired. By *offering pairs of problems* (fig.24a) one can diagnose the children's insight in the properties of the operations. This kind of items is particularly informative if the children's answers are being compared with those on numerically similar series of bare sums. One can then observe great differences. This child gets correctly the sums with support problems, yet not those without support (fig.24b), while another succeeds just on the sums without support(fig.24c), yet not on those with a support added (fig.24d).

Test 4.4 (8)

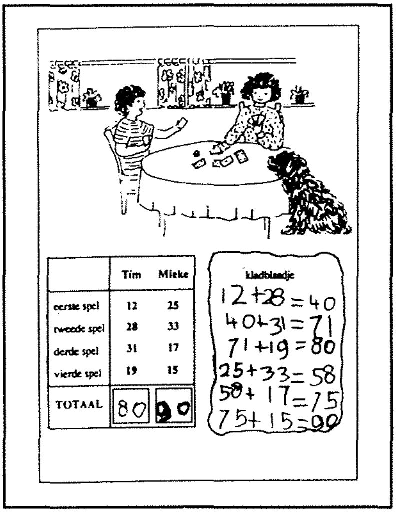


fig.25a

Test 4.4 (8)

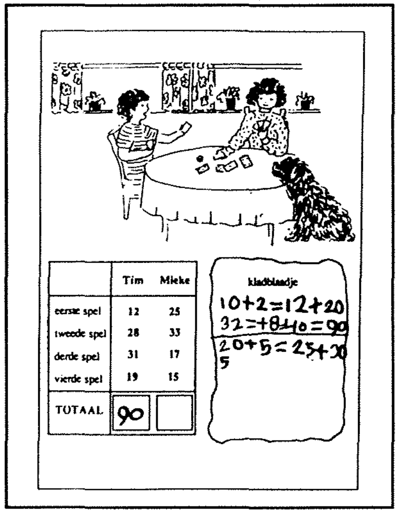


fig. 25b

Test 4.4 (8)

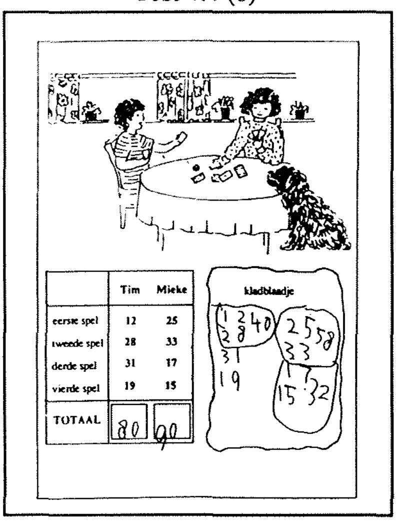


fig.25c

Test 4.4 (8)

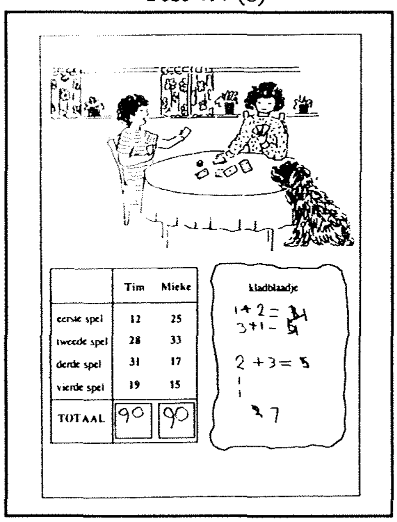


fig. 25d

The large amount of information about children's strategies provided by the preceding test items is even raised by the use of *'pieces of scrap paper'*. You can see the pieces pictured on a test sheet (fig.25a). The test itself is about two children playing a game: at the end the scores of both of them are to be calculated. In order to do it the students may use the scrap paper. Some students leave them empty, others let it be known that they don't need them, but a quite a number of pieces show traces of the solving strategies.

Take, for instance, this student (fig.25a). Obviously the student added the numbers one after the other. The next (fig.25b) started the same way (10 + 2 = 12, . . .), yet arriving at 40, he obviously noticed that the result was 90. The third (fig.25c) took pairs of numbers together, though with respect to the second column not in the simplest way. The same holds for the fourth (fig.25d). This scrap of paper also shows that additions are being made from the left to the right, that is, first adding the tens, and after adding the units, correcting the tens.

One more test item with scrap paper (which has been omitted for a moment) (fig.26a). The length of the long bridge being 48 metres, it is asked: 'How long is the short bridge?' The answer of this student is: '18.' But what does such an answer reveal, except that it is wrong? Look, however, how revealing such a simple piece of scrap paper can be.

As in the case of the own productions it is evident that pieces of scrap paper provide both a rich treasure of data on the children's procedures and very useful instructional material.

Test 4.4 (13)

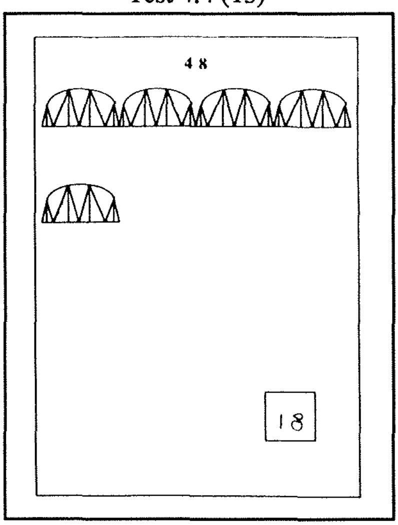


fig. 26a

Test 4.4 (13)

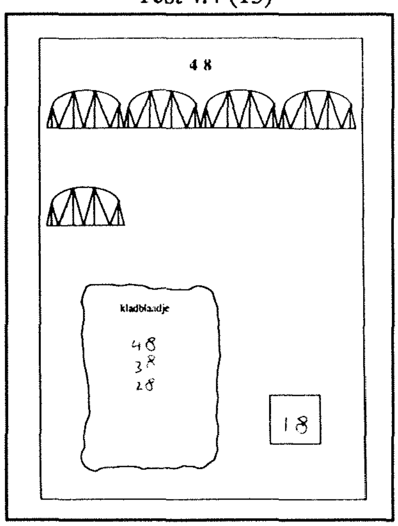


fig. 26b

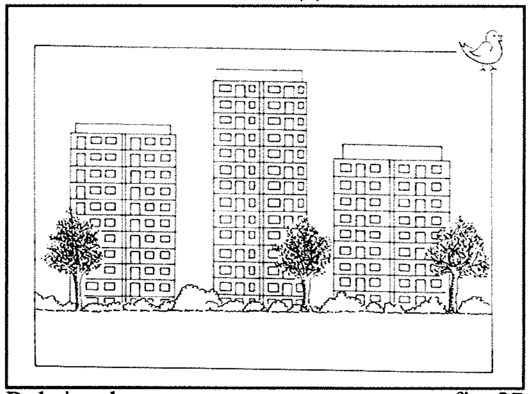
4.9 A most revealing example to conclude with

After this review of a variety of test items, which extend the limits of testing, I am going to conclude by analysing one test in its totality. Indeed, after the isolated examples of items, the total picture of one test can even better prove how revealing a good test can be. Moreover, this particular example will show that written tests in the classroom environment can be useful even if given to young children.

The test I am going to deal with was developed in order to find out what knowledge and abilities with regard to number children possess when entering grade school. In our country, apart from preparatory activities at the kindergarten level, systematic arithmetic/mathematics instruction starts at the first grade (= 6 years old children).

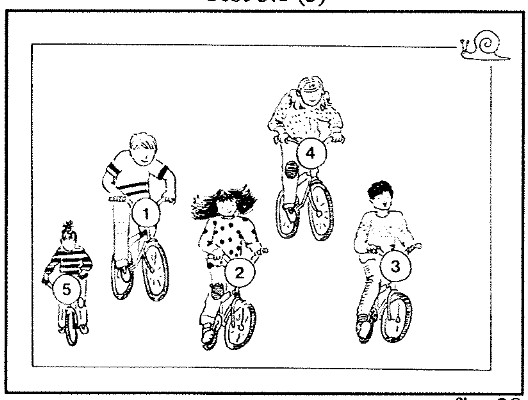
The test was administered after three weeks in the first grade, so the children had got nearly no arithmetic/mathematics instruction, nor could they read or write, nor had they had any earlier classroom experience in doing written tests. For these very children the test had been developed.

Test 3.1 (1)



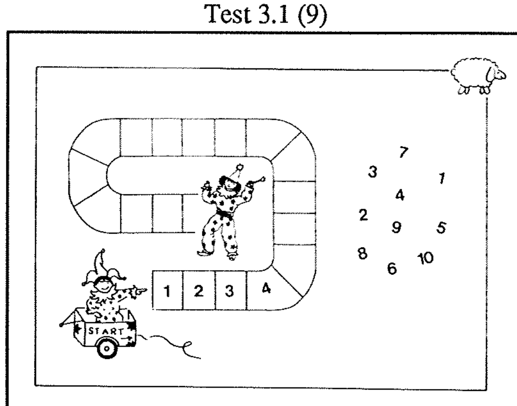
Relational concepts fig. 27

Test 3.1 (5)



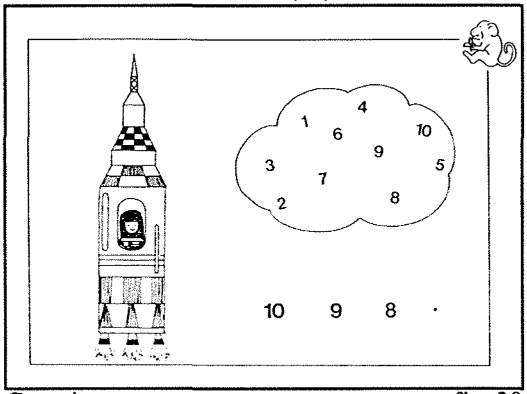
Knowledge of symbols fig. 28

Test 3.1 (9)



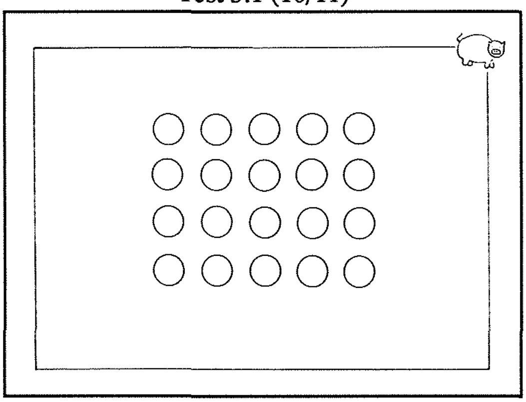
Counting sequence fig. 29

Test 3.1 (16)



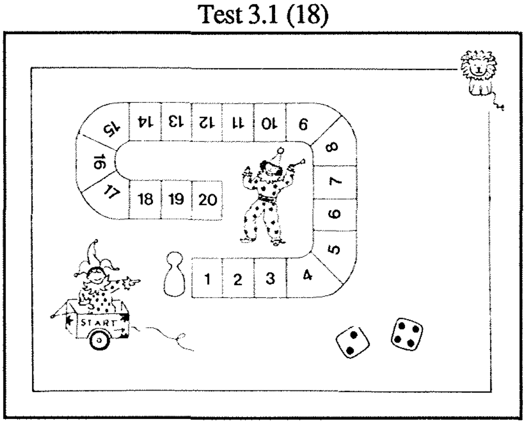
Counting sequence fig. 30

Test 3.1 (10/11)



Resultative counting fig. 31

Test 3.1 (18)



Adding within a context (countable) fig.32

The test consisted of the following parts:

- relational concepts;
- knowledge of symbols;
- the counting sequence;
- resultative counting;
- operations: addition and subtraction.

The part 'relational concepts' included the concepts 'highest', 'smallest', 'thickest', and 'most'. For instance, this item (fig.27) aims at 'highest'. The subjects are asked to put a cross at the highest building.

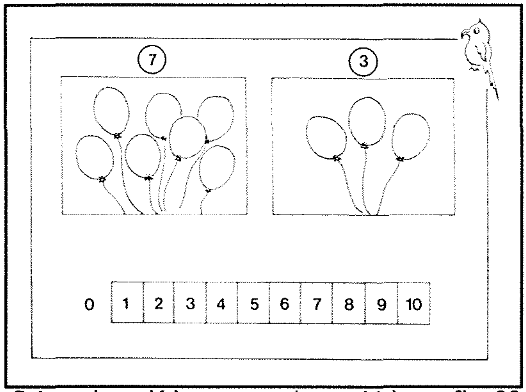
'Knowledge of symbols' was operationalised by trying to make sure whether the children knew the numerals 3, 5, 10, 14. For instance, the second item (fig.28) asks: 'Put a cross at number 3!'

With respect to the counting sequence it was tested whether the children knew which numbers followed on 4 and 7 and which preceded 4 and 8. To indicate the follower the picture of an incomplete goose board was shown (fig.29), which was to be completed, that is, the subjects had to cross out the number that was to turn up. The model for finding the predecessor was the way they are counting when launching a rocket (fig. 30).

For resultative counting (fig.31) the subjects had to colour 2, 5, 7, 9 marbles, respectively.

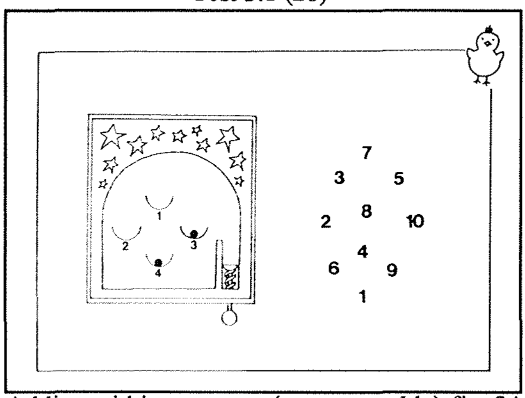
Adding and subtracting were tested in a context, rather than by formulas. This happened on two levels: tasks where the sheet shows sets of countable objects, and tasks where the quantities are given by numerals. The first two items (fig.32 and 33) show the 'countable' variant of adding and subtracting. For adding the question is 'where to put the counter?', for subtracting, 'how many balloons have been sold?' The next two items (fig.34 and 35) show the 'non-countable' variant. For adding the question is 'how many points together?', and for subtracting the subjects must indicate the number of florins they will keep after buying the goggles.

Test 3.1 (24)



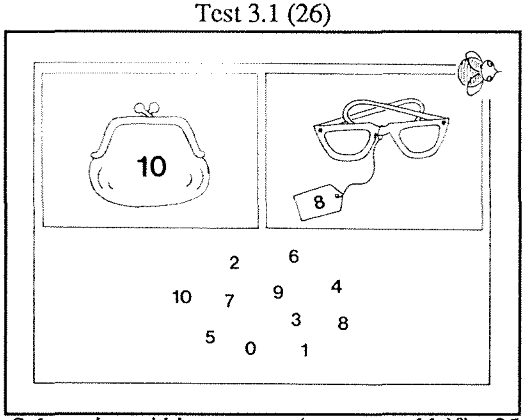
Subtracting within a context (countable) fig. 33

Test 3.1 (20)



Adding within a context (non-countable) fig. 34

Test 3.1 (26)



Subtracting within a context (non-countable)fig. 35

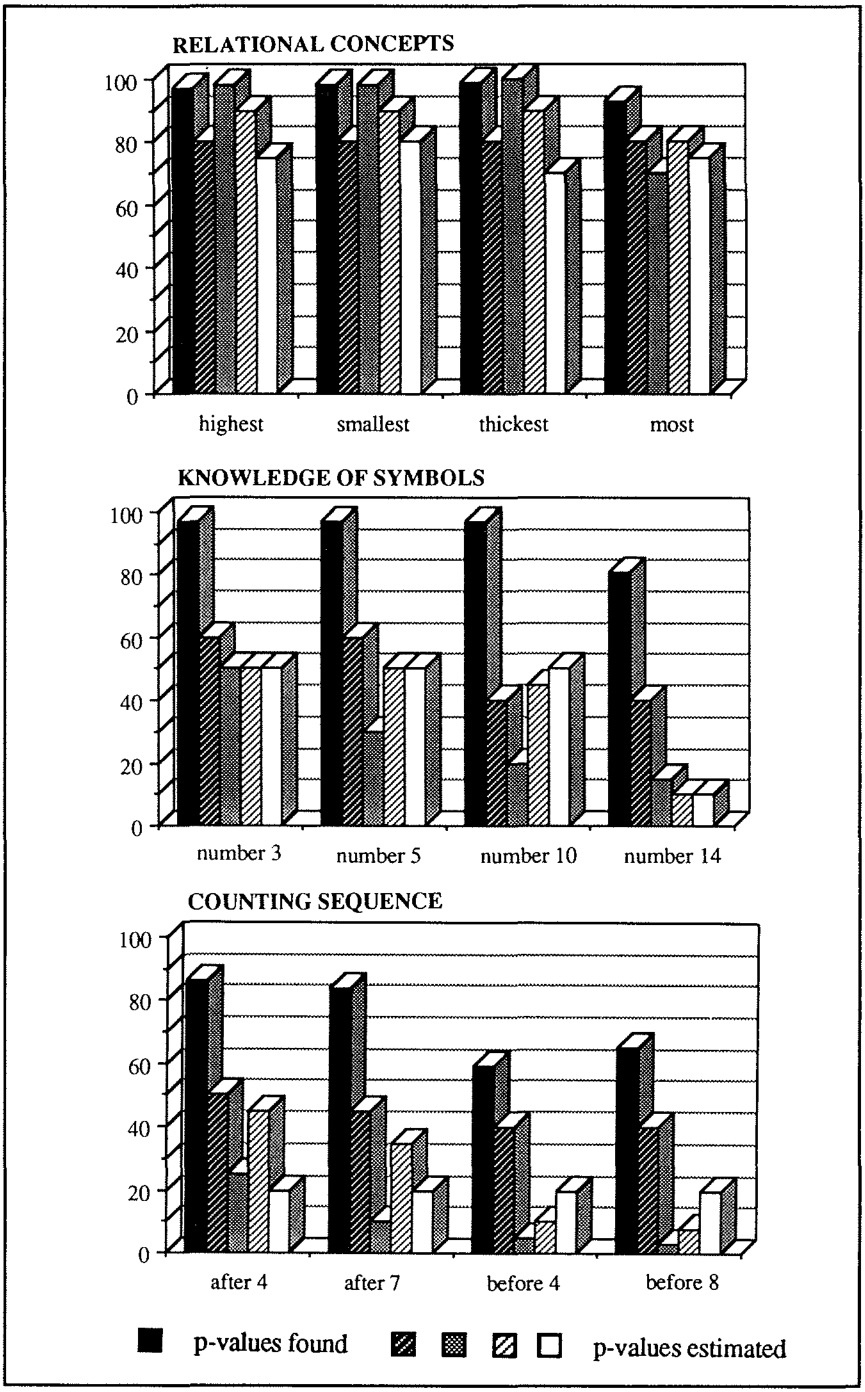

RELATIONAL CONCEPTS
100
80
60
40
20
0
highest
smallest
thickest
most
KNOWLEDGE OF SYMBOLS
100
80
60
40
20
0
number 3
number 5
number 10
number 14
COUNTING SEQUENCE
100
80
60
40
20
0
after 4
after 7
before 4
before 8
p-values found
p-values estimated

fig. 36

The test was administered to 22 first grade classes. The population was quite heterogeneous: rural and city schools, schools with many foreign children and schools with a great majority of Dutch children, schools which are using a realistic textbook and school which are using a traditional textbook. 441 children took the test. In each class the teacher administered the test according to an instruction that prescribed all details. Before summarising the results I would ask you to make estimates for yourself. What do you think children are able to at the start of grade 1, that is, at the age of six?

In The Netherlands we posed the same question to four groups of four to five people working as teachers in primary education, as counsellors, or as teacher trainers. The figures 36 and 37 reproduce their estimates.

They expected full mastery of the relational concepts (fig. 36). Their expectations with regard to knowledge of symbols were much lower– about half of the children would know the numerals up to 10. With regard to the counting sequence they were still lower – a fourth would be able to answer the questions. About the same would hold for counting with regard to 7 and 9 (fig. 37). The lowest were the expectations with regard to adding and subtracting. With 'countable' objects they would somehow succeed, but this percentage was estimated to sharply decrease for the 'non-countable' variant. In particular for subtraction the estimates were very low.

I don't know whether in your American situation you would agree with these estimates, but for the Netherlands they were far off the mark. The relational concepts were mastered by almost all subjects (fig. 36), and the same holds for the knowledge of the numbers up to 10. At 'knowledge of the counting sequence' the great majority knew the follower. With the predecessor it is different: only half of the children succeeded. Resultative counting up to 10 was mastered by almost all children (fig. 37). At adding and subtracting within a context children succeeded very well with respect to adding small numbers, in particular, in the case of countable objects. Even tasks with number symbols were mastered by about half of the subjects. With subtractions the scores were lower, and here was no significant difference between 'countable' and 'non-countable'.

Anyway it is evident that children at the start of the first grade possess quite a bit of numerical knowledge and abilities. Obviously they were grossly underestimated, at least by our judges.

For us it was most revealing that this was discovered by a written test in a classroom environment. In other words, even in realistic arithmetic/mathematics instruction, tests may have a future.

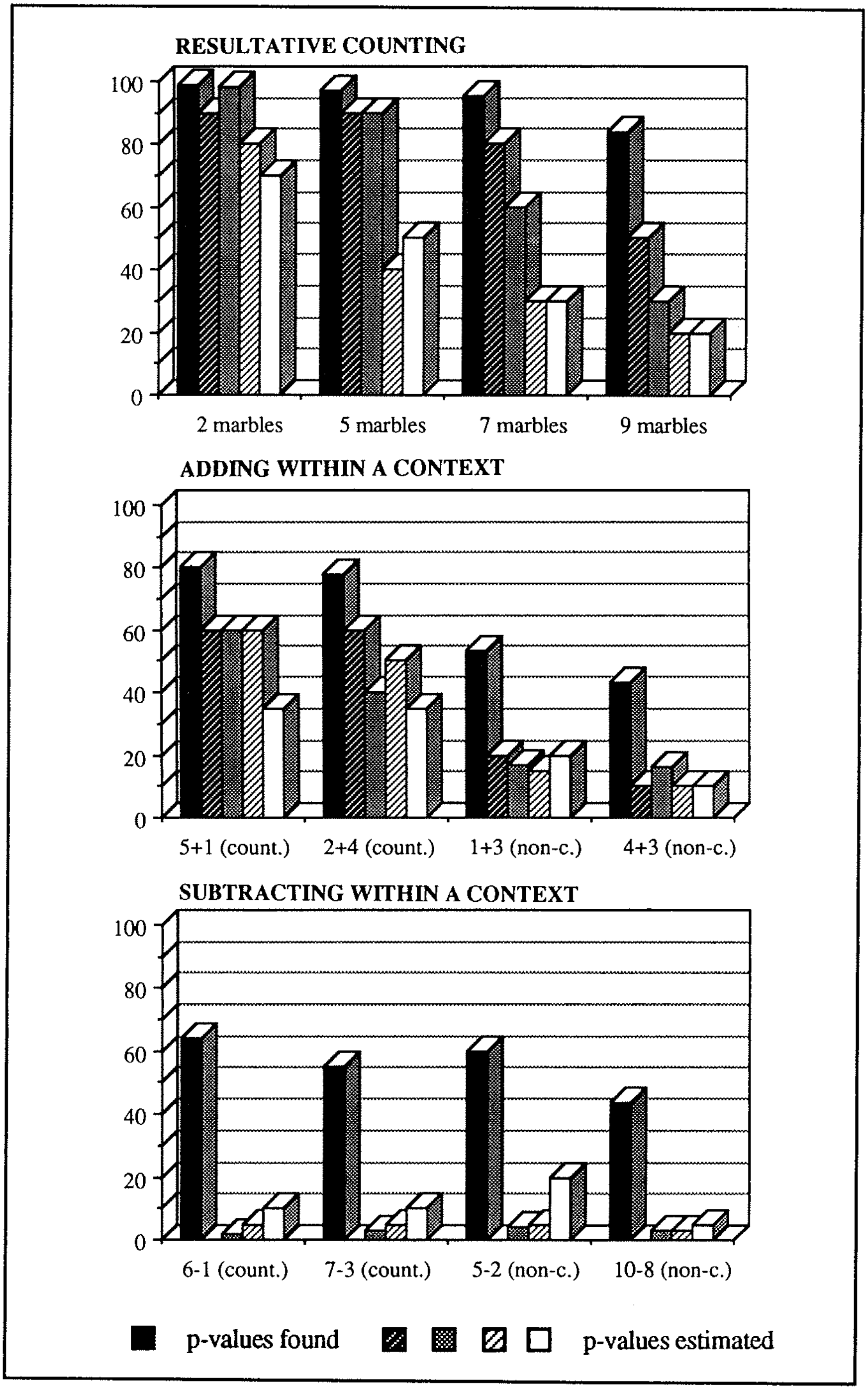

RESULTATIVE COUNTING
100
80
60
40
20
0
2 marbles
5 marbles
7 marbles
9 marbles
ADDING WITHIN A CONTEXT
100
80
60
40
20
0
5+1 (count.)
2+4 (count.)
1+3 (non-c.)
4+3 (non-c.)
SUBTRACTING WITHIN A CONTEXT
100
80
60
40
20
0
6-1 (count.)
7-3 (count.)
5-2 (non-c.)
10-8 (non-c.)
p-values found
p-values estimated

fig. 37

Notes

1. Heesen, H., D. Stelitski, A. van der Wissel: *Schiedamse Rekentest*, Groningen, Wolters Noordhoff, 1971.
2. Lange Jzn, J. de: *Mathematics, Insight and Meaning*, Utrecht, OW&OC, 1987.
3. The MORE-project is a collecgive researchproject of the Researchgroup OW&OC and the Department of Educational Research, both of the State University of Utrecht. The project is supported by a grant from the Dutch Foundation for Educational Research (SVO - 6010).

5 Realistic Geometry Instruction

Koeno Gravemeijer

5.1 Introduction

The present contribution deals with a kind of geometry instruction which differs largely from the well-known deductive geometry such as taught at secondary levels in most countries. We are pleading for 'realistic geometry', if not as a replacement then at least as a valuable preparation for more formal geometry. This sketch of realistic geometry will also be used to clarify some aspects of realistic instruction theory.

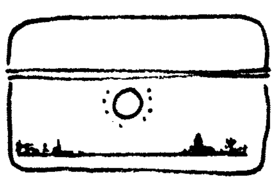
5.2 Examples of realistic geometry

Let us illustrate the idea of realistic geometry by a few examples.

1 What do you do if you want to take a photograph of a large company and you see that you will not get everybody on the picture?
...... You will step back.

Why does the photographer, standing farther away, get more people on the picture? It looks so obvious that hardly anybody would ask the question. However, the answer requires a geometric interpretation of the situation. The next examples are to show the intriguing character of geometric problems, which are suggested by everyday life experience.

2 You're travelling a long way by train.
The train is moving through a moonlit landscape.

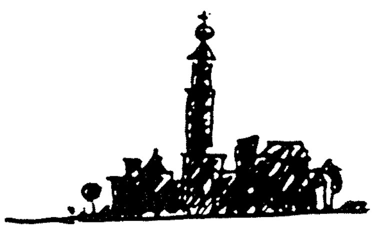


Whatever the speed of the train may be,
the moon is catching up with you, or so it seems.
>> What is the cause of this effect?

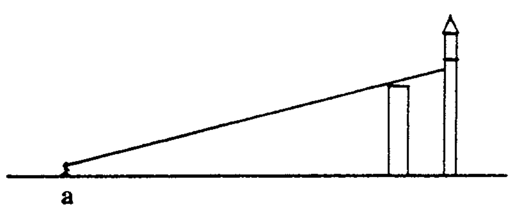
3 *How comes a mirror interchanges right and left, while conserving above and below?*

4 *If you are driving through a long city street, huge buildings in the background look like sinking down behind the lower ones in the foreground.*
>> What is happening?

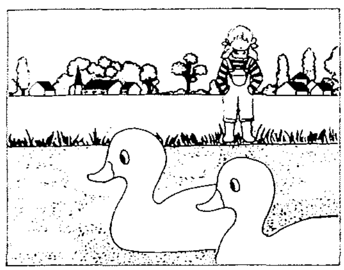
Or, to put the last question in another way: when, in our flat Dutch landscape, we approach the silhouette of a town at the straight horizon, the church towering high above its surroundings lowers, in order eventually to hide behind the other buildings.



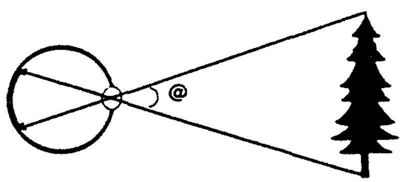
A sideview tells what's happening. Let us take a church and a hotel in front of it. A spectator at point a, looking at the town, what part of the church does he see?



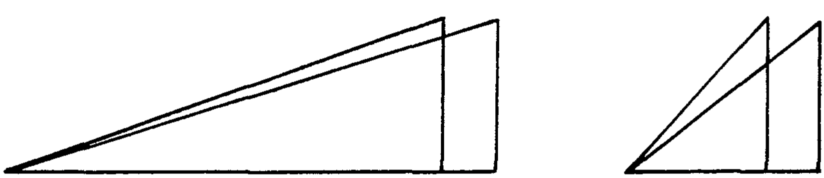
We draw his (straight) vision line to see how it changes if he starts moving and how the visible part of the church gradually diminishes. So to him the church seems sinking behind the other building. However, is it sinking or shrinking? In fact, it is growing bigger as one gets closer, but the nearer building seems to grow faster than that at the background. Let us elaborate a bit further on this point.
Any object, as it looks bigger it is in fact closer. This is so obvious that one does not even think about it. But let us investigate the cause of this effect. How big an object shows, depends on what we may call its vision angle (@). Its size is determined by the size of the object as well as by the distance between object and observer. The bigger the distance, the smaller the angle. So the distance determines the scale at which objects are seen.



What looks bigger is in fact closer

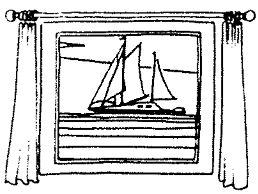


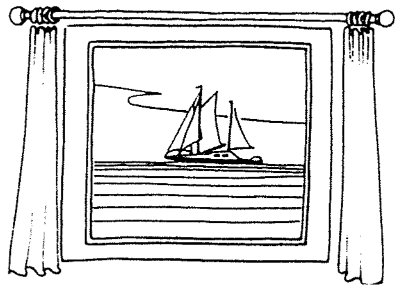
Let us return to the hotel and the church tower.
Then we may reason as follows. Far away (a) both buildings are seen at about the same scale. But as one gets closer (b), the difference between the scales is getting ever larger.



a. vision angle: far away *b. vision angle: close by*

This relative change of scale can also be observed
when approaching a window: the closer one gets, the wider the view.





One might say: 'Nice problems, but is it geometry, and even more, is it mathematics?' Let us answer the last question first, and then come back to the first. What mathematics means depends on what one chooses it to be: a ready made system or an activity.

4.1 Mathematics and mathematics instruction

In realistic mathematics instruction we agree with Freudenthal (1971) who sees mathematics as an activity, not unlike a mathematician's activity, that is, an activity of solving problems, looking for problem, and organising or mathematising a subject matter.

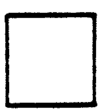
> This can be matter of reality which has to be organised according to mathematical patterns if problems from reality are to be solved. It can also be a mathematical matter, new or old results, of your own or others, which have to be organised according to new ideas, to be better understood, in a broader context, or by an axiomatic approach.
> A great part of mathematical activity today is organising. We like to offer the results of our mathematical activity in a well organised form where no traces betray the activity by which they were created. (Freudenthal 1971; 413 - 414.)

Freudenthal fights the way to teach mathematics by implementing the results of the mathematical activities. Whereas in history mathematics started with real life problems to evolve more general and more formal ideas, instruction starts with the formal system, in order to present applications afterwards. He calls this an anti-didactical inversion. It deprives the students of the opportunity to experience mathematics as a mathematical activity.
To challenge this educational tradition one would start with real life problems and stimulate mathematising as the main learning principle. Mathematising may enable students to reinvent mathematics, rather than to absorb preconstructed mathematics. With a reference to the history of mathematics, Freudenthal supports this reinvention principle as a source of inspiration for curriculum designers.

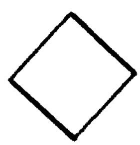
Next to mathematising Freudenthal mentions 'looking for problems' as a mathematical attitude. Realistic geometry is a marvellous field to develop and practise this reflective attitude. Reflecting on practice has played a key role in the development of geometry. Geometry started with solving practical problems. Geometry was wisdom about a craft before people started studying it, out of curiosity rather than on behalf of profitable applications, though afterwards results of this activity proved applicable as well. Egyptians are told to have used the (3, 4, 5) triangle to construct right angles[1]. By the Greek tradition Pythagoras was credited with having raised geometry to the level of a liberal art, that is, an art exercised by free citizens rather than by craftsmen. His disciples' work led, among others, to trigonometry, useful in surveying, navigation and astronomy.
This is only one example of processes where solutions of theoretical problems, dating from an earlier period, became practical tools later on. Informal experiential knowledge one once started with, became a matter of reflection to create higher level knowledge. So if we take the point of view of mathematics as an activity we may reach to the conclusion that reflecting on practical geometric problems is mathematics, which leaves us with the first question: 'Is it geometry?'. To answer it let us look at Van Hiele's analysis of geometry instruction.

4.2 Van Hieles level-theory

Geometry instruction as understood by Van Hiele (1973, 1985) largely diverges from the tradition. Although he studied instructional problems at a time when deductive geometry still belonged to the core curriculum of Dutch secondary instruction, his analysis of the didactical problems arising with this kind of geometry instruction is still useful to show us the importance of less formal introductions.
Van Hiele states a communication gap between teacher and student, which he illustrates by means of their interpretations of the geometrical concept 'rhomb'. 'This figure is a rhomb', he says, may mean quite different things to teachers and students.



square



square seen as a rhomb

The student might recognise the shape and associate it with the name 'rhomb'. For students who recognise a rhomb this way it may be a hard thing to see a square as a rhomb, unless this square is placed in another position. To a mathematician, as well as to a mathematics teacher, the label 'rhomb' has quite another

meaning. It is a rather large collection of properties and relations such as:
- it is a polygon;
- all sides are equally long;
- it is a parallelogram;
- opposite sides are parallel;
- the diagonals are perpendicular, and so on.

Because of such properties the teacher will range a square among the rhombs. Yet even a rough sketch of a figure suggesting equal and parallel sides will be accepted as a rhomb. Teacher and student differ with regard to their referential framework. These conceptual differences block the communication. The same words do not have the same meaning for both of them. So different referential frameworks effect different conceptual levels, and the only way to tackle the problem is to have the needed referential framework constructed at the lowest conceptual level.

As a matter of fact Van Hiele distinguishes three conceptual levels in the process of learning geometry. At the teacher's conceptual level words like 'rhomb', side', 'angle', 'square', etc. establish junctions in a framework where each of them is constituted by a bunch of properties. This is what Van Hiele calls *the second level*. Yet there is no framework like this at *the first, or ground level*, where the labels are still connected to concrete experiences and perceptual objects.

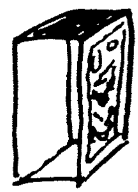
On the *third level*, however, the relations themselves become object of thinking: properties of relations and the connections between properties are settled, which makes it possible to construct a logical system.

According to this description, former Dutch deductive geometry in secondary education can be said to have started at the third level, rather than at the first, with concrete problems and concrete activities. To be sure, what's concrete depends on the students' factual knowledge. In other words, the three levels should not be absolutised since each subject allows for its own three levels. Even though concepts like point, line and angle may be concrete to secondary school students, knowledge about them may be third or second level as soon as, rather than traditional geometry, the subject matter in question is orientation in space.

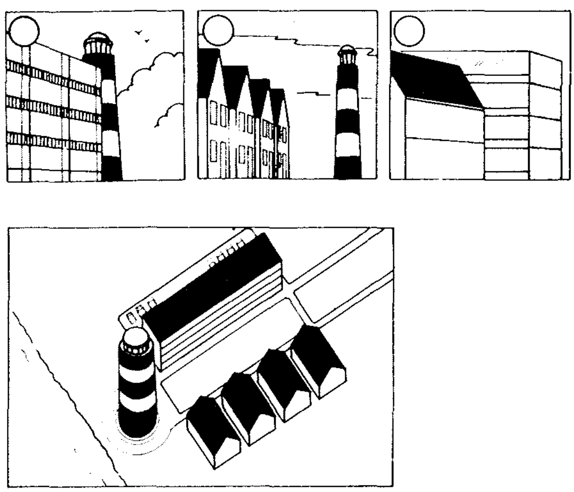
4.3 Realistic instruction theory

Nevertheless the Van Hiele levels can help establishing the macro-structure of a course. To this end let us rephrase the level description as did Treffers (1987) who distinguished the *intuitive phenomenological level*, the *locally descriptive one*, and the *level of subject matter systematics*. Next to this macro-structure a micro-didactical structure is needed. It originates from Freudenthal's (1983) didactical phenomenology and the reinvention principle[2], and it is defined by the relation between reflection and transition to higher micro-levels, where, as said above, *the solutions to the problems of an earlier period become tools at a later one*. The driving force is a reflective attitude, which may be developed by real-

istic geometry instruction. We may exploit the intriguing force of geometry to stimulate this aspect of mathematical attitude. Realistic geometry gives us a splendid opportunity to develop it thanks to the vast amount of informal geometrical knowledge within young children's grasp. An impression of this kind of geometry instruction may be mediated by sketching a few activities proposed in a realistic textbook series for primary school (Gravemeijer; 1983).
First, it starts with exploring what looks like taking pictures: children are equipped with frames of matchboxes which function as cameras. A student gets the task to take a picture of the teacher that shows her completely. By experimenting he will find out that it counts how the camera is hold,

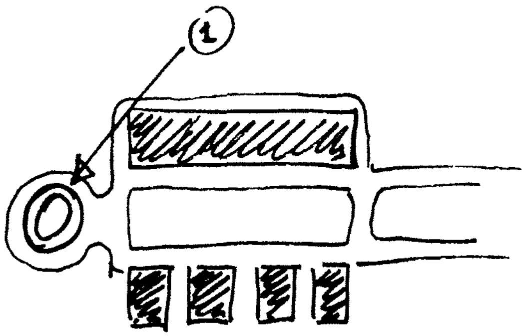
this way  or that way 

and especially, whether you stand close by or farther away.
Next time a birds eye view of a village is presented, together with some pictures of the same spot. At each picture the students will have to answer the same question:

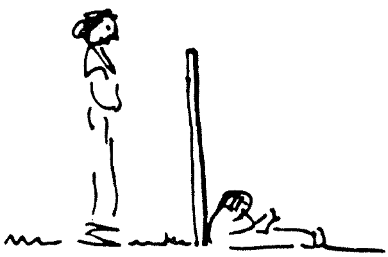


Where stood the photographer?

A mock up of the situation is available to the children who tackle this problem, so they can check their ideas, or use the mock up to try and find out. When reconstructing the viewpoint of the photographer, the students may implicitly reconstruct his vision lines.



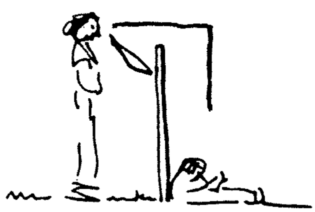
Later on we may introduce this kind of lines more explicitly with problems like the following:



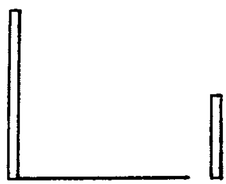
Uncle Bill *Marc*

Can Uncle Bill see Marc?

Onno (7 years old), confronted with this problem, reasoned 'No, he can't, since he cannot look this way', and drew a curved line.



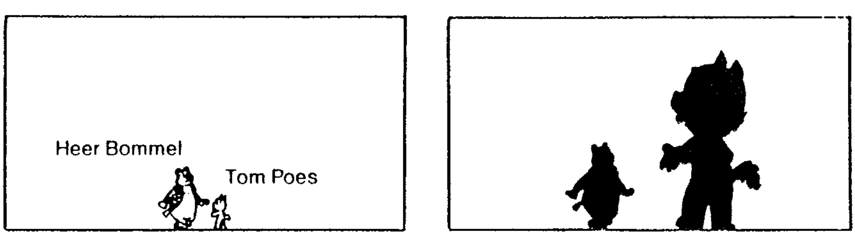
'Neither can he look through the wall', he added and drew a line to represent a lightbeam reflected by the wall.
Another task might be the following:



Draw the shadow of the second rod.

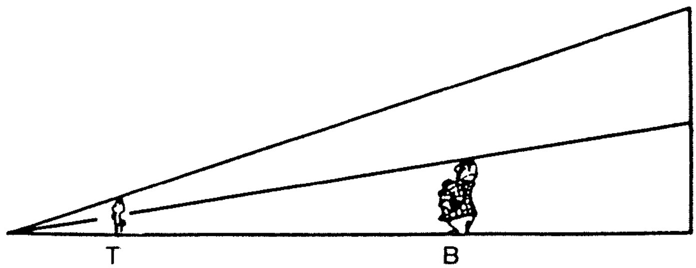
Shadows can be compared by drawing parallel lines, but one can also reason that in the sunlight twice as small a rod gives twice as small a shadow. The combination of these two insights is an intuitive base for understanding invariable ratios in similar triangles.
With other light sources things can be different, as experienced in a comic by Big Bear Bommel and Tom Cat.

The dwarf Barribal uses shadows to look bigger than he is.
Explain how it is being done.

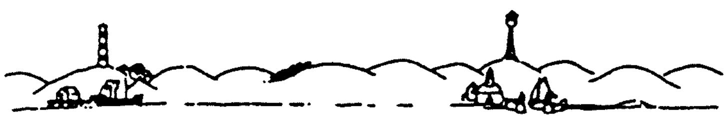


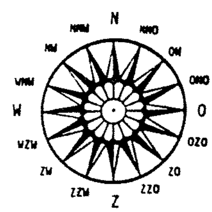
Though Bommel is bigger than Tom Cat; their shadows don't behave accordingly.

A sideview may be used to explain the size of the shadows.



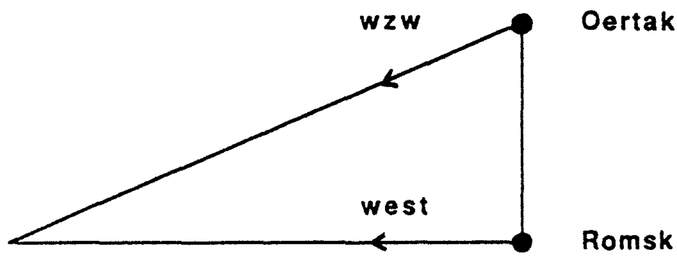
Over some period, the shadow-model will evolve into the more sophisticated triangle model, where no references to shadows and so on are needed to know about the fixed relation between the shape of a right triangle and the ratio of its sides.
This can be elaborated in all kind of problems. For instance:





The coast of Soldaria stretches exactly north-south.
In the north is the harbour Urtak, and a bit south of it the harbour Romsk.
One day the coast guard of Urtak notices an emergency signal west-south-west.
From Romsk the same signal is seen right in front of the coast, that is, west.
Make a drawing of the situation (a map).
The distance Urtak-Romsk is 3 km. How far off the coast is the ship?

One doesn't need trigonometry to solve this problem. Just draw what you know, and find out the distance by measurement and ratio.



This rough sketch may give some idea of a possible learning sequence on a topic in realistic geometry. It intertwines a few learning strands: getting familiar with the model of the right triangle, visualising geometric problems, and viewing situations from different angles.

4.4 Conclusion

A realistic geometry programme for the primary school may stand as an example for realistic mathematics instruction, which in turn fits into the theoretic framework of Van Hiele levels as a global macro-structure, Freudenthal's view on reinvention for micro-structures, and didactical phenomenology as an indication of how reality can be used to start learning processes. Treffers (1987) a posteriori formulated this theoretical framework of realistic mathematics instruction and analysed the main characteristics of educational programmes and textbook series developed according to this approach (see also Streefland; 1990a). These characteristics will be illustrated once more by the geometric problems presented above.

4.4.1 The dominating place occupied by context problems, both serving as source and as field of application of mathematical concepts.

We may refer to the context problems on orientation and shadow which preluded on the idea of fixed ratios in similar triangles.

4.4.2 Broad attention paid to (the development of) situation models, schemas and symbolising.

The shadow model proved to be a useful bridge between intuitive notions on shadows and mathematical relations between the shape of a right triangle and the ratio of sidelengths.

4.4.3 Large contributions to the course by the children's own productions and

constructions, which lead them from informal to formal methods.
The constructions appeared in solving procedures such as Onno's reasoning. We also indicated different level of solving procedures. As regards free productions, we refer to examples by Streefland (1990b) and Van den Heuvel (1990).

4.4.4 The interactive character of the learning process.
Interactivity will be exemplified by the cooperation between two 11 years old girls who were working on the Oertak problem. They drew a map of the two harbours and the direction of the ship, and started measuring. But then one of them expressed her doubts on this method. Would the arbitrary choice of a distance for the two towns on the map affect the result? After a short discussion they realised that it would not; because of the similarity of all possible triangles one might draw, the answer would always be the same - an important learning result which without this interaction would hardly have been made explicit.

4.4.5 The intertwining with extern learning strands.
This intertwining becomes obvious as soon as the knowledge about angles, triangles and ratio is used in graphs and in calculus.

Notes.

1. It is a fairy tale, the oldest source of which dates just a century back; though the Egyptians knew and applied a lot of practical geometry, there is not any indication that they knew at least this special case of the so-called Pythagorean theorem. (Freudenthal, 1986).
2. See Gravemeijer (1990).

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